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AMTI

THE MATHEMATICS TEACHER

THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA

The Association of Mathematics Teachers of India (AMTI) was started in 1965 for the promotion of efforts to improve Mathematics education at all levels. A major aim of the Association is to assist practising teachers of Mathematics in schools in improving their expertise and professional skills. Another important aim is to spot out and foster Mathematical talents in the young. The Association also seeks to disserminate new trands in mathematics education among parents and public. Other activities of the Association include consultancy services to schools in equpping the Mathematics section of their libraries, in organising children's Mathematics clubs and fairs, in setting up teacher centres in schools, in conducting Mathematics laboratory programmers, in holding practical tests in Mathematics in assisting children in participating investigational projects.

The Association holds "National Mathematical Talent Search Competition" annually and organises Orientation Courses, Seminars and Workshops for teachers and courses for talented students. A national conference is held annually in different parts of the country for teachers to meet and deliberate on important issues of Mathematics education. Innovative teacher award has been instituted to give public recognition to enterprising and pioneering teachers of Mathematics for which entries from teachers are invited.

An award for contributions to the Mathematics Teacher relating to History of Mathematics in the context of mathematics education has been instituted by Prof. R.C. Gupta.

"The Mathematics Teacher (India)" (MT) is the official quarterly journal of the Association and is issued twice a year. It has been approved for use in schools and colleges of education by the Government departments of education in many States. Besides MT the Association also brings out Junior Mathematician (JM), three issues in a year, especially for school students in English and Tamil.

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EDITORIAL

In this issue of "The Mathematics Teacher" some articles are included. Some articles are based on papers presented at conferences. I again request the readers to send articles on some new results or their experiences of teaching mathematics. One can write on some notable historical facts from ancient Indian Mathematics.

I am thankful to Dr. D.V. Shah for compiling the "Mathematics News" section. Solutions to the two problems given in vol.57(3,4) are given and also two new problems are posed. Please send your solutions to the editor directly on: mjbhatt9@yahoo.com.

Turning Errors into Insights: A Reflective Approach to Mathematics Learning

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Introduction: *Mathematics often evokes anxiety in students—largely due to a fear of making mistakes. This fear, rooted in conceptual gaps, contributes to what is commonly referred to as "math phobia". To overcome this, there is a growing need to shift the educational focus from simply correcting errors to understanding the reasons behind them.*

An error-analysis and error-tracing approach offers a powerful tool in this context. By closely examining students' mistakes, educators can uncover underlying misconceptions, recognize thinking patterns, and provide timely interventions. This not only strengthens students' foundational understanding but also fosters a more open and growth-oriented learning environment—where errors are viewed as learning opportunities rather than failures.

The key objective of this initiative is to demonstrate how analysing and learning from errors can significantly improve mathematical comprehension. By transforming mistakes into meaningful learning moments, students are encouraged to develop critical thinking skills, build conceptual clarity, and gain confidence in problem-solving. The

approach also bridges the often-seen gap between procedural fluency and conceptual understanding.

At the heart of this study lies the hypothesis that systematic error analysis can reduce recurring conceptual errors and enhance reasoning abilities. Students engage with specially designed worksheets embedded with intentional errors, which prompt deeper reflection and logical analysis. These activities not only support metacognition but also promote a resilient and inquisitive mindset.

Key Causes of Mathematical Difficulties

A major contributor to underperformance in mathematics, particularly in middle school, is the prevalence of conceptual errors. While younger students often make computational mistakes, older students struggle with abstract, application-based problems due to underlying conceptual gaps—fuelling math anxiety.

Another challenge is the lack of number sense and overreliance on mechanical, pen-and-paper methods. This limits logical reasoning, slows problem-solving, and increases both computational and conceptual errors, especially under time pressure.

Short-term retention of core concepts further compounds the problem. Students often fail to internalize key ideas, resulting in poor knowledge transfer and repeated re-teaching. With an emphasis on test scores over conceptual depth, they rarely engage in sustained practice, weakening long-term understanding.

Additionally, students tend to rely heavily on familiar problem types. When presented with unfamiliar or reframed questions, especially in high-stakes exams, their performance drops significantly due to limited adaptability.

Finally, a lack of exposure to diverse strategies prevents students from integrating concepts or exploring efficient problem-solving paths—undermining their readiness for advanced mathematical thinking and STEM-related pursuits.

Procedure of Action Research

The action research adopted a structured approach to error analysis and tracing, aimed at strengthening conceptual understanding in mathematics. It began with identifying and classifying conceptual errors, particularly those that often went unnoticed or overlapped with computational mistakes. Recognizing these deeper gaps was a crucial first step.

A diagnostic phase followed, using tools like unit tests, assignments, and classwork to uncover students' problem-solving patterns. Regular, structured sessions were then introduced, where students collaboratively analysed worksheets containing common errors. These peer discussions promoted group learning, self-reflection, and critical thinking.

The sessions encouraged students to explain their reasoning and evaluate others' thought processes, helping them internalize key concepts more effectively. Teacher guidance played a vital role throughout, ensuring

timely clarification of misconceptions and reinforcing accurate mathematical thinking.

Part 1: Identifying and Addressing Common Mathematical Errors

A diagnostic review conducted in authentic classroom settings provided a comprehensive picture of students' misconceptions. Errors were categorized based on the nature of the misunderstanding, and targeted corrective strategies were developed to address them. The focus was on concept reinforcement rather than rote correction, ensuring students not only fixed errors but understood the "why" behind their corrections.

Representative samples of common errors, linked misconceptions, and conceptual insights were documented and formed the foundation for subsequent instructional design.

The erroneous samples	The correct method	Concept misunderstood at focus
Solve: $-\lvert-4\rvert$ Answer: $-[-4]=+4$	$-\lvert-4\rvert=-4$	The symbol of 'absolute value of a number is mistaken as 'brackets'.
Find : $-5-5$ Answer: $(-5)(-5)=(+10)$	$-5-5=(-10)$	Sign rule applied blindly

Subtract (-25) from (-8) Answers: $(-8)-25=-33$ $(-25)-(-8)=-17$ $(-25)-8=-33$	$(-8)-(-25)=(+17)$	The sign of the integer has become insignificant and this is confused with operational sign. Also language is not appropriately comprehended (.i.e Subtract from)
Simplify $(3-2+5)=3-7=(-4)$	$(3-2+5)=+6$	The BODMAS applied & addition done first.
$\frac{8}{9}+\frac{21}{16}=\frac{1}{3}+\frac{7}{2}=\frac{8}{5}$	$\frac{8}{9}+\frac{21}{16}=\frac{128}{144}+\frac{189}{144}=\frac{317}{144}$	During the addition of fractions, cancellation is done.
$\frac{4}{3}+\frac{7}{3}+5=5\frac{11}{6}$	$\frac{4}{3}+\frac{7}{3}+5=\frac{11}{3}+5$ $=5+\frac{11}{3}=8\frac{2}{3}$	During the addition of fractions, adding all denominators, as well as all numerators. Also, writing a mixed fraction with improper as its part.
$\frac{4}{7}\times\frac{6}{14}=\frac{8+6}{14}=\frac{14}{14}=1$	$\frac{2}{7}\times\frac{6}{7}=\frac{12}{49}$	During the multiplication of fractions, LCM of denominators taken (as in addition). Cancellation is missed out.

$\begin{array}{r} 568 \\ 17\overline{)86156} \end{array}$	$\begin{array}{r} 5068 \\ 17\overline{)86156} \end{array}$	In the long division process, zero is not written/ missed out when a number can't be divided at a particular stage.
$2^3 + 2^5 + 2^4 = 2^{12}$	$2^3 + 2^5 + 2^4$ $8 + 32 + 16 = 56$	Wrong usage of law of exponent
$\frac{3}{8} \div \frac{5}{4} = \frac{8}{3} \div \frac{5}{4} = \frac{8}{3} \times \frac{4}{5}$	$\frac{3}{8} \div \frac{5}{4} = \frac{3}{8} \times \frac{4}{5} = \frac{3}{2} \times \frac{1}{5} = \frac{3}{10}$	Conversion of division sentence to multiplication is inappropriately carried out.
$6x^0 + (6x)^0 = 1 + 6 = 7$ $6x^0 + (6x)^0 = 1 + 1 = 2$	$6x^0 + (6x)^0$ $= (6 \times 1) + 1 = 7$	Wrong usage of law of exponent
$(-2)^4 = -2^4$ $3^2 = 2^3$ $5^3 = 15$	$(-2)^4 = +16$ $-2^4 = -16$	The base and the exponent of an exponential number not considered meaningfully. (Interchanging the base and exponent, Neglecting the sign carried by the base, multiplying base to exponent to find the value of the exponential number)

By how much is 10 greater than 7 Answer 10+7=17	10-7=3 10 is 3 greater than 7	Linguistic misconception on 'By how much is…..' How much is….
Find the greatest 2-digit number which is divisible by 2, 4, 6 Ans: The Greatest number means HCF (H stands for highest). So HCF=2. Its multiple=98 is the answer.	Greatest 2-digit number which is divisible by means; the number we find is bigger and it is LCM LCM=12 The multiple of 12 is 96 is the answer.	The meaning of HCF and LCM and their properties are confused.
Find $(5x-3y)-(-2x-y)z$ Answer: $(5x-3y+2xz-yz)$	$(5x-3y+2xz+yz)$	The negative sign outside the bracket is not multiplied to the binomial entirely. Only partial multiplication is done. While explaining, replacing the variables with numbers might help.
I spent three-fourth of the pocket money. If one-third of it is spent on movie, one- sixth on book and the rest on dress, what part of the entire pocket money is spent on dress?	Multiply three fourth to one third to get fraction w.r.t movie. Also, multiply three fourth to one sixth to get fraction w.r.t book. Add the two answers. Subtract the	The inner meaning of 'fraction of a whole' 'fraction of a fraction' is not clearly understood. The pictorial representation using circle or bar model

Answer: Add one-third and one- sixth. Subtract the sum from three-fourth.	sum from three fourth. Answer: Three eighth.	method might work better.
Factorise $\left(\frac{y}{2}\times x\right)+\left(\frac{x}{2}\right)$ Answer: $\left(\frac{x}{2}\right)(y)$	$\left(\frac{x}{2}\right)(y+1)$	The distributive property is not visualised/ understood clearly.
Find using property 1234x 99+1234 Ans:1234 (90+9)+1234 =(1234x90)+1234x9+ 1234	1234 (99+1)=1234x100 =123400	The distributive property is not visualised/ understood clearly. Visual explanation using word problems as examplesof sharing will help
Find $\left(\frac{m}{2}\right)+m-6m$ Answer: $-5m+\left(\frac{m}{2}\right)$	$m\left(-5+\frac{1}{2}\right)=\left(\frac{-9m}{2}\right)$	The basic algebra is not very deeply absorbed. (Meaning of terms and expressions and factors in algebra)

Convert into improper $-7\frac{5}{8}$	$-[(7\times 8)+5]=-(61)$ $Ans;\frac{-61}{8}$	The sign of the rational number is mistaken for whole number part of the mixed form.

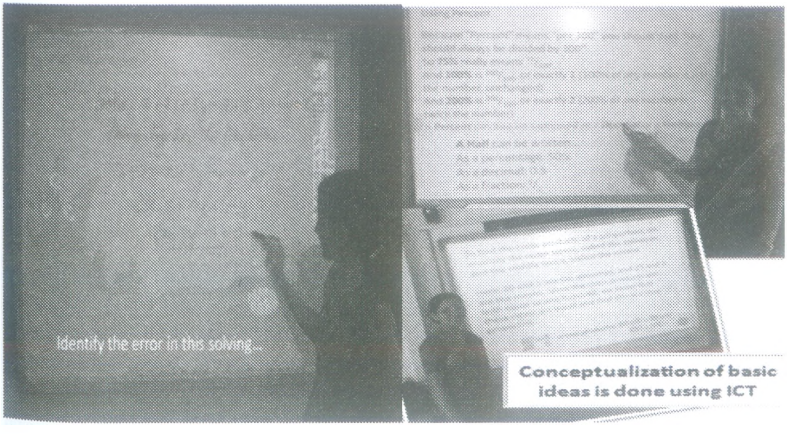
Answer: $(-7\times 8)+5=(-51)$ $Ans;\frac{-51}{8}$		
$(x+y)^2=x^2+y^2$	$(x+y)^2=x^2+y^2+2xy$	The algebraic identity is not appropriately used.
$\sqrt{x+y}=\sqrt{x}+\sqrt{y}$	$\sqrt{x+y}=\sqrt{x+y}$	
$\frac{x}{y_1+y_2}=\frac{x}{y_1}+\frac{x}{y_2}$	$\frac{x_1+x_2}{y}=\frac{x_1}{y}+\frac{x_2}{y}$	Common denominator is allowed, but not common numerator.
$\frac{1}{x+y}=\frac{1}{x}+\frac{1}{y}$	$\frac{1}{x+y}=\frac{1}{x+y}$	
$(\sin x)^2=(\sin^2 x^2)$	$(\sin x)^2=(\sin^2 x)$	

These corrections were not aimed at simply fixing the answer, but at building durable conceptual clarity.

Part 2: Building Conceptual Strength through Structured Error Comparison

To deepen understanding, students were given opportunities to compare correct solutions with incorrect ones, allowing them to observe how misconceptions diverged from accurate mathematical reasoning. This comparative method sharpened their logical thinking and illuminated the subtleties of problem-solving.

Error-based worksheets—categorized by difficulty (easy, standard, challenging) and frequency of error types—were introduced progressively. These worksheets functioned not only as practice tools but also as platforms for independent reflection, small group discussion, and pattern recognition. Over time, students demonstrated increased fluency in identifying conceptual errors and applying corrective strategies independently.



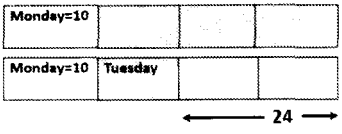
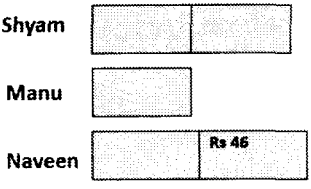
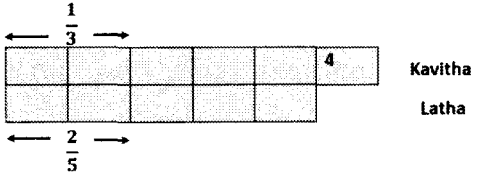
	Erroneous Exercise **Category: Easy**	**Correct Method**
1	Find d: $5 - \frac{2d+7}{9} = 0$ $5 = \frac{2d+7}{-9}$ $5 \times -9 = 2d + 7$ $-45 = 2d + 7$ $-45 + 7 = 2d$ $-38 = 2d$ $-38 \div 2 = -19 = d$	$5 = \frac{2d+7}{+9}$ $5 \times 9 = 2d + 7$ $45 = 2d + 7$ $45 - 7 = 2d$ $38 = 2d$ $38 \div 2 = 19 = d$
2	Divide $(6x^2 + 10x)$ *by* $2x$ $(6x^2 + 10x) \div 2x$ $= \frac{6x^2+10x}{2x}$ $= \frac{6x^2+10\cancel{x}}{2\cancel{x}}$ Cancelling 10 and 2 by 2, we get $6x^2 + 5$ Alternate approach leading to error $\frac{6x^{\cancel{2}} + 10x}{2\cancel{x}}$ Cancelling 6 and 2 by 2, we get $3x + 10x = 13x$	$\frac{6x^2+10x}{2x}$ $= \frac{6x^2}{2x} + \frac{10x}{2x}$ $= 3x + 5$

	Erroneous Exercise **Category: Standard**	**Correct Method**
1	Solve $\frac{6p-9}{3} = \frac{2p+9}{5}$ Cross Multiply $5 \times 6p - 9 = 3 \times 2p + 9$ $30p - 9 = 6p + 9$ $30p - 6p = 9 + 9$ $24p = 18$ $p = \frac{18}{24} = \frac{3}{4}$	$\frac{(6p-9)}{3} = \frac{(2p+9)}{5}$ Cross Multiply $5 \times (6p - 9) = 3 \times (2p + 9)$ $30p - 45 = 6p + 27$ $30p - 6p = 27 + 45$ $24p = 72$ $p = 72 \div 24 = 3$
2	Find $\frac{(2x+1)(3x)+(4x+2)}{2x+1}$ $\frac{\cancel{(2x+1)}(3x) + (4x + 2)}{\cancel{2x+1}}$ Cancelling $2x + 1$ by $2x + 1$, we get $3x + 4x + 2$ $7x + 2$	$\frac{(2x+1)(3x)+(4x+2)}{2x+1}$ $\frac{(2x+1)(3x)+2(2x+1)}{2x+1}$ $\frac{(2x + 1)(3x + 2)}{(2x + 1)}$ Cancelling $2x + 1$ by $2x + 1$, we get $(3x + 2)$

	Erroneous Exercise **Category: Difficult**	**Correct Method**
1	Find p If $x = p - 2$ and $\frac{x}{p} - \frac{x+1}{p} = 1$ Substitute $x = p - 2$	$\frac{x}{p} - \frac{x+1}{p} = 1$ $\frac{p-2}{p} - \left(\frac{p-2+1}{p}\right) = 1$

	$\frac{p-2}{p}-\frac{p-2+1}{p}=1$ $\frac{p-2}{p}-\frac{p-1}{p}=1$ $\frac{p-2-p-1}{p}=1$ $p-2-p-1=p$ $p=-3$	$\frac{p-2}{p}-\left(\frac{p-1}{p}\right)=1$ $\frac{p-2-(p-1)}{p}=1$ $p-2-p+1=p$ $p=-1$
2	Factorise $9(a-b)^2-100(x-y)^2$ $[3(a-b)]^2-[10(x-y)]^2$ $(3a-3b)^2-(10x-10y)^2$ $(3a-3b+10x\ 10y)\ X$ $(3a-3b-10x-10y)$	$9(a-b)^2-100(x-y)^2$ $3^2(a-b)^2-10^2(x-y)^2$ $[3(a-b)]^2-[10(x-y)]^2$ $(3a-3b)^2-(10x-10y)^2$ $[3a-3b-(10x-10y)]$ $\times[3a-3b+(10x-10y)]$ $[3a-3b-10x+10y]$ $\times[3a-3b+10x-10y]$

SINGAPOREAN BAR MODEL METHOD	
1	Kumar reads 10pages of a book on Monday, $\frac{1}{3}$of the remaining pages on Tuesday. If he has 24 pages left to complete the book, find the total number of pages in the book. Solution: Total=10+3 equal bars 2 equal boxes=24 ; This implies 1 Box= 12

	Hence, Total = 10+ (3x 12) = 46 pages 
2	Shyam has twice the amount as Manu. Naveen has Rs 46 more than Manu. If all three have a total of Rs 210, find the amount each has.  4 boxes+46 = Rs 210 4 boxes=210 - 46 = Rs 164 1 box = 41 Hence, Shyam: Rs 41x2= Rs 82 Manu: Rs 41 Naveen: Rs 41+Rs 46= Rs 87
3	Kavitha is 4 yr elder to Latha and the sum of their ages is 44 yr. Also, one-third of Kavitha's age equals two-fifth of Latha's age. Find their ages.  10 boxes+4 = 44 yr 10 boxes= 44-4 = 40 yr 1 box = 4 yr Hence, Kavitha's age = 5 boxes+4= (5x4)+4 =24 yr Latha's age= 5 Boxes= 5x 4 =20 yr Verification: One-third of 24= 8 and Two-fifth of 20= 8

Part 3: Fostering Conceptual Clarity through the "Learn from Mistakes" Approach

Moving beyond procedural correction, a "learn from mistakes" philosophy guided students toward meaningful mathematical engagement. This involved prompting students to reflect on their errors, understand the logic behind mathematical operations, and explore the interconnectedness of concepts.

To support this shift:

- Visual strategies, such as the Singapore Bar Model, were introduced to break down complex relationships, especially in arithmetic and algebra.
- Students engaged in verbal reasoning and peer discussions, explaining their steps and justifying corrections using mathematical principles.
- Frequent "number talks" "Mathematical discourse" cultivated mental math fluency, estimation skills, and curiosity about numerical relationships.

Through these strategies, students began to view errors not as setbacks but as entry points for deeper learning.

E.g 1 - Consider the multiplication of fractions and the simplification by cancellation. *Find* $\frac{7}{15} \times \frac{24}{7}$

A student answered it wrongly as follows

$$\frac{7}{15}\times\frac{24}{7}=\frac{7}{3}\times\frac{8}{7}=\frac{8}{3}$$

The error is caused due to cancelling the numerator by a certain number and the denominator by another. If the concept is taught as shown below, it is better understood.

The simplification/ cancellation means dividing both numerator and denominator by the same number and this finally results in simplest form.

$$\frac{7}{15}\times\frac{24}{7}=\frac{7}{7}\times\frac{24}{15}=1\times\frac{24\div3}{15\div3}=\frac{8}{5}$$

E.g 2 - *Find* $\frac{12}{16}+\frac{1}{4}$

The student answered wrongly as follows. He used the process of multiplication (Cancellation) instead of addition. $\frac{12}{16}+\frac{1}{4}=\frac{3}{16}+\frac{1}{1}=\frac{3}{16}$

If the solving is taught as follows, it is comprehendible.

$$\frac{12}{16}+\frac{1}{4}=\frac{3}{4}+\frac{1}{4}=\frac{3+1}{4}=\frac{4}{4}=1$$

Re*ason* $\frac{12}{16}=\frac{12\div4}{16\div4}=\frac{3}{4}$

E.g 3 - The distance between Sun and Earth is $1.496\times10^{11}m$ and the distance between Earth and Moon is $3.84\times10^{8}m$ .What is the distance between Sun and Moon during the lunar eclipse. (Earth comes between Sun and Moon)

The student answered wrongly as follows.

$$1.496\times10^{11}+3.84\times10^{8}$$
$$=1.496\times10^{11}+3840\times10^{11}$$
$$=10^{11}\times(1.496+3840)$$
$$=10^{11}\times3841.496=10^{8}\times3.841496$$

The explanation which may help clear the confusion is

$$3.84=3.84\times10^{0}=38.4\times10^{-1}=384\times10^{-2}=3840\times10^{-3}$$
$$3.84\times10^{8}=3840\times10^{-3}\times10^{8}=3840\times10^{5}$$

Teaching a concept more than one way, students were exposed to a richer math environment and this opened opportunity to dig into the content. They chose methods as per their preference.

A student solved the above problem as follows, in an alternate/ better way.

$$.496\times10^{11}=1.496\times10^{3}\times10^{8}=1.496\times1000\times10^{8}=1496\times10^{8}$$
$$.496\times10^{11}+3.84\times10^{8}=1496\times10^{8}+3.84\times10^{8}$$
$$=10^{8}(1496+3.84)=10^{8}\times1499.84$$

Visualization Techniques and Enhancing Conceptual Understanding

To make abstract ideas more tangible, students were encouraged to visualize concepts through drawings, diagrams, and simplified breakdowns of problems. Contextualizing concepts, particularly in word problems, enabled students to form mental models that enhanced retention and comprehension. ICT tools were also used to supplement these efforts and provide dynamic representations of mathematical ideas.

Crucially, student-initiated error analysis was positioned as a stepping stone to teacher-facilitated correction. When recurring misconceptions were noted across the class, concepts were re-taught using new approaches tailored to student needs.

Part 4: Cultivating Logical Reasoning and Critical Thinking through Erroneous Worksheets

Open sessions dedicated to error analysis nurtured critical thinking by encouraging students to analyze both their own work and that of peers. Discussions around "what went wrong" and "how to fix it" helped students move from surface-level corrections to deeper, reasoned understanding.

To support these explorations, verbal and visual aids such as annotated examples, guided hints, diagrams, and concept charts were integrated into instruction. These tools helped students draw connections, consider alternative strategies, and approach unfamiliar problems with greater flexibility, confidence, and creativity.

Part 5: Evaluating the Impact of Error Analysis and Creating a Reference Framework.

The effectiveness of error analysis was measured through qualitative tracking of misconceptions across multiple evaluations, rather than relying solely on short-term test scores. Student progress was documented through worksheets, assessments, and reflective assignments, offering a clear view of learning trajectories and long-term improvement.

This analysis led to the development of a structured reference guide for teachers—a practical tool, mapping common misconceptions to targeted strategies. The guide now serves as a resource for refining instructional plans, adjusting unit goals, and embedding error-based learning into routine practice.

Key Insights from Error Analysis

The qualitative review of student work—conducted through both teacher-led and peer-assisted error analysis—yielded critical insights into learning patterns and persistent misconceptions. Diagnostic sessions using carefully designed erroneous worksheets showed that many conceptual errors were not isolated incidents but reappeared across topics, especially in Grades 7 and 8.

By systematically categorizing errors and tracking their frequency, clear trends emerged. Algebra and linear equations were the most error-prone areas, followed by exponents, mensuration, and commercial mathematics. These recurring difficulties often stemmed from foundational conceptual gaps that extended across multiple chapters, underscoring the need for a layered and interconnected teaching approach.

These findings prompted a re-evaluation of instructional planning. Teachers were able to adapt unit plans, revisit challenging concepts using alternative strategies, and introduce preventive interventions earlier in the learning cycle.

Outcomes and Reflections

The implementation of structured error analysis led to significant gains in conceptual clarity. Students showed a marked decrease in recurring mistakes, increased confidence, and a more engaged attitude toward mathematics. The impact was not limited to short-term performance—students retained concepts more effectively and continued to demonstrate improvement in follow-up assessments conducted months later.

The reflective, student-centred approach supported the development of metacognitive awareness, allowing learners to recognize, evaluate, and correct their own thinking. As students became more articulate in explaining mathematical procedures and their reasoning, the gap between procedural fluency and conceptual understanding began to narrow.

While most students responded positively, a few high-achieving learners found the frequent focus on errors to be occasionally confusing, suggesting the importance of differentiated scaffolding to suit varying proficiency levels.

Challenges and Considerations

Despite its clear benefits, the approach was not without its challenges. Developing effective error-based activities required considerable planning time, and the success of implementation relied heavily on skilled facilitation to avoid inadvertently reinforcing misconceptions. Additionally, because the growth observed was primarily qualitative,

traditional assessments were insufficient to capture the full impact. Long-term observation and reflective tools proved more effective for measuring progress.

Conclusion

Error analysis emerged as a transformative strategy for enhancing conceptual understanding and cultivating resilient mathematical thinkers. By shifting the classroom culture to one where mistakes are valued as learning opportunities, students became more thoughtful, confident, and reflective in their mathematical approach.

For educators, the process provided a responsive framework for refining instruction and addressing evolving student needs with greater precision and empathy. The insights and outcomes of this action research not only affirmed the value of reflective practices in math education but also offered a sustainable model for deep, meaningful, and lasting learning.

Typical Problems and Solutions.

- **SN Maitra**

(The author is a former Head of Mathematics Dept. at National Defence academy, khadakwasla Pune)

Ed: The Original article is edited to simplify so that school children can understand. Only natural number solutions are considered. Those who know about irrational and imaginary solutions can try to get if possible.

Problem -1 If the sum of the squares of the first n natural numbers is 385, find the value of n.

Solu : The formula is $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$ and given

$\sum n^2 = 385$

$\therefore \frac{n(n+1)(2n+1)}{6} = 385$

$\Rightarrow n(n+1)(2n+1) = 385 \times 6 = 2310.$

It is clear that this is a cubic equation which will not be easy to solve. We look at it from a different angle. The left side has 3 factors. The first is 'n'., second is one more than the first and the third is one more than the double of the first. It we can express right side number, is 2310 in such 3 facton, we get the solution. Now

$$2310 = 10 \times 231 = 10 \times 11 \times 21$$

$$= (10)(10+1)(2\times 10+1)$$

hence $n = 10$

Problem -2 If $a^3 + b^3 = 152$ and $ab = 15$, find the value of $a + b$.

Solu : We know

$a^3 + b^3 = (a + b)^3 - 3ab(a + b)$

Substituting the given values and $a + b = x$ .....(i)

$$152 = x^3 - 45x$$

$\therefore x^3 - 45x - 152 = 0$ .....(ii)

let $x = m^{1/3} + n^{1/3}$ .....(iii)

cubing both side,

$$x^3 = \mathrm{m} + \mathrm{n} + 3m^{1/3}n^{1/3}x$$

➡ $x^3 - 3m^{\frac{1}{3}}n^{\frac{1}{3}}x - (m + n) = 0$ .....(iv)

Comparing with eq: (ii)

$(m + n) = 152$ .....(v)

$mn = (15)^3$ .....(vi)

Using the value of (v) & (vi) in the identity

$(m-n)^2 = (m+n)^2 - 4mn$, we get

$(m-n) = 98$ .....(vii)

From (v) and (vii) $m = 125, n = 27,$

From eq. (iii), $x = (125)^{1/3} + (27)^{1/3} \therefore x = 8$

$\therefore$ by (i) $a + b = 8.$

[Alternatively, equation (ii) can be written as, $x^3 - 45x = 152$

$\therefore x(x^2 - 45) = 152$ ..... (A)

now as done earlier, find two factor of 152 and see which value of x satisfier both sides.

$x(x^2 - 45) = 152 = 2\,x\,76$

If $x = 2,\ x^2 - 45 \neq 76$ and if $x = 76,\ x^2 - 45 \neq 2,$ so it will not work. Similarly checking other pairs of factor, we get

$x = 8,\ x^2 - 45 = 64 - 45 = 19$ and $8\,x\,19 = 152$

$\therefore a + b = 8,$ .....(B)

As $x = 8, x - 8$ *is a factor of eq.* (ii)*Hence*

$x^3 - 45x - 152 = x^3 - 8x^2 + 8x^2 - 64x + 19x - 152 = 0$

$= x^2(x-8) + 8x(x-8) + 19\,(x-8) = 0$

$= (x-8)\,X(\ x^2 + 8x + 19) = 0$

$\therefore\ x^2 + 8\,x + 19 = 0$ is another factor. Solving this quadratic equation by usual method, we can see two imaginary roots ie.

$x_2 = -4 + \sqrt{3}\,i$, $x_{3=-4-}\sqrt{3}\,i$. So, there are two other values of a+b (besides 8) which are imaginary --------- editor]-

Problem -3 If $a^3 - b^3 = 98,\ ab = 15$, find the values of

$a^2 + b^2$ and a+b. (a,b ∈N).

Solu : We have

$(a-b)^3 = a^3 - b^3 - 3ab\,(a-b)$

using the given values,

$(a-b)^3 = 98 - 45\,(a-b)$ let a$-b = x$ .....(i)

$\therefore x^3 + 45x = 98$ ..... (ii)

$\therefore x(x^3 + 45)x = 98 = 2 \times 49 = 7 \times 14$

$x = 2, x^2 + 45 = 49$ and other option do not satisfy the equation. Hence $x = a - b = 2$ is one value. Also $x - 2$ is a factor of

$x^3 + 45x - 98$. By simple calculation we get another factor

$x^2 + 2x + 49$

$\therefore x^2 + 2x + 49 = 0$ will give two other imaginary values of x (and hence of $a - b$).

Now $a^3 - b^3 = (a - b)(a^2 + b^2 + ab)$

$\therefore 98 = (2)(a^2 + b^2 + 15)$

$\therefore a^2 + b^2 = 34$ *and* $(a + b)^2 = a^2 + b^2 + 2ab \rightarrow (a + b) = 8.$

(Note: taking $a + b = -8$ will give -ve numbers) -ed

Problem - 4 If $x + y + z + t = 10$, $x^2 + y^2 + z^2 + t^2 = 30, xy = 2$ *and* $zt = 12,$ *find the values of* x, y, z *and* t

Solu : Let $x + y = a$ *and* $z + t = b \therefore a + b = 10 \ldots\ldots\ldots\ldots (i)$

$$x + y = a \Rightarrow (x + y)^2 = a^2$$

$$\Rightarrow x^2 + 2xy + y^2 = a^2$$

$$\Rightarrow x^2 + y^2 = a^2 - 4 \ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

$$z + t = b \Rightarrow (z + t)^2 = b^2$$

$$\Rightarrow z^2 + 2zt + t^2 = b^2$$

$$\Rightarrow z^2 + t^2 = b^2 - 24 \ldots\ldots\ldots\ldots\ldots\ldots (iii)$$

As $x^2 + y^2 + z^2 + t^2 = 30$ given, using (ii) & (iii),

$$a^2 - 4 + b^2 - 24 = 30 \Rightarrow a^2 + b^2 = 58 \ldots\ldots\ldots\ldots (iv),$$

Using (i) and (iv) in the identity $(a + b)^2 = a^2 + 2ab + b^2$, we get $ab = 21$

Using different possible sets of 21, the reader can find the values of

x, y, z *and* t *satisfying all the equations.*

Editorial Note :

The author has also given two more problems of the similar nature. We give them for the reader to solve.

Problem – 5 *if* $a^3 + b^3 = 35$ *and* $a^2 + b^2 = 13$, *find* $a + b$.

Problem–6 *if* $a^3 + b^3 + c^3 = 197$, $a^2 + b^2 + c^2 = 45$, *and* $abc = 40$, *find* $a + b + c$.

Hunt for the solutions of Pell's equation

Dr.Devbhadra V.Shah

Introduction:

In number theory, Pell's equation falls in the category of Diophantine equations named after the Greek mathematician *Diophantus*. Diophantine equations are equations for which integer solutionsare desired. Such an equation may have no solution, a finite number of solutions or an infinite number of solutions. The famous Pythagorean equation ! " is also a Diophantine equation.

In 1909, mathematician *Thue* proved the following important theorem:

Theorem: Let $f(x,y) = a_n x^n + a_{n-1}x^{n-1}y + \cdots + a_1 xy^{n-1} + a_0 y^n; n \geq 3$ then for an integer $M \neq 0$, the equation $f(x,y)$. has either no solution or only a finite number of solutions in integers.

This result is in contrast to the situation in which n = 2. In this case, if we consider F(x, y) = $x^2 - Dy^2$, then for $N \neq 0$, the equation $x^2 - Dy^2 = N$ may have infinitely many integral solutions! Specifically, the misnamed *Pell's equation* is used to refer any Diophantine equation of the form $x^2 - Dy^2 = N$ where **D** and **N** are fixed integers and we are looking for integers x and y that satisfy the equation.

If **D** is a perfect square, say D = q^2, then equation $x^2 - Dy^2 = N$ becomes $x^2 - (qy)^2 = N$. This factors as $(x - qy)(x + qy) = N$. Since x, y, and q are all integers, the left-hand side of this equation must be product of two factors of **N**. Thus $x^2 - Dy^2 = N$ has only finite number of solutions. It could also happen that the equation has no solution.

In the equation $x^2 - Dy^2 = N$, if D < **0**, say D = $-d$, , then the equation becomes $x^2 + Dy^2 = N$ This can be written in the form

$\frac{x^2}{(\sqrt{N})^2} + \frac{y^2}{\left(\sqrt{\frac{N}{D}}\right)^2} = 1$, which has the form of an ellipse (a closed curve). But any ellipse can have only finite number of integer points (x, y) on the circumference. Thus, $x^2 - Dy^2 = N$ (where D < 0) has only finite number of integer solutions. We thus assume that D > 0.

Again, it can be easily observed that $x^2 - Dy^2 = N$ can be written as $\frac{x^2}{(\sqrt{N})^2} + \frac{y^2}{\left(\sqrt{\frac{N}{D}}\right)^2} = 1$, which is a Hyperbola. Thus, the integer solutions occur whenever the curve passes through a point whose and y coordinates are both integers.

Thus, we consider N to be any nonzero integer and the integer D to be positive and non-square only. This condition is helpful because it leaves open the possibility of infinitely many solutions in positive integers x and y and !.

We now give formal definition of Pell's equation.

Definition: A *Pell's equation* is an equation of the form $x^2 - Dy^2 = N$, where D is a positive non-square integer and N is a nonzero integer, for which we attempt to find integer solutions x and y.

Some natural questions which arise at this stage are…

(a) Is there any algorithm or fixed procedure which can decide whether a given Pell's equation has a solution?

(b) Is there any algorithm for solving explicitly any given Pell's equation?

Our goal here in this article is not to describe the present status regarding these questions. It suffices to say that a complete answer to question (a) is not yet available. There is no complete procedure to decide whether any Pell's equation has a solution or not. Also, question (b) is unanswered.

Historical encounters:

Pell's equations have been of interest to mathematicians for centuries. There is perhaps no other equation that has influenced the development of number theory as much as Pell's equation. Pell'sequation has a long and interesting history. This very simple Diophantine equation has attracted mathematicians throughout the ages. There is historical evidence that methods for solving the equation were known to the Greeks some 400 years before the beginning of the Christian era. We find the reference of Pell's equation in a problem given by Archimedes in 200 BC. Indeed, there is very strong evidence that it was known to Archimedes, as the *Cattle Problem*. In *Arithmetica*, *Diophantus* asked for rational solutions to equations of the type $x^2 - Dy^2 = 1$ For the case $D = m^2 + 1$ he gave the integral solutions $x = 2m^2 + 1, y = 2m$.

The equations $x^2 - Dy^2 = \pm 1$ have been studied by several Indian mathematicians also. In 8th century BC. *Baudhayana* noted that $x = 577, y = 408$ is a solution of $x^2 - 2y^2 = 1$ and he used the fraction $\frac{x}{y} = \frac{577}{408}$to approximate the value of $\sqrt{2}$.

The equation $x^2 - Dy^2 = 1$ attracted attention of early mathematicians. Such equations and assertive rules, without any proof for calculating their solutions, spread to India more than a thousand years before they appeared in Europe, starting with *Brahmagupta* (598 – 668). He developed the *chakravala method* in 628 AD to solve Pell's equation in his *Brahma Sphuta Siddhanta* in 628, about a thousand years before Pell's time. This method could generate infinitely many solutions from an initial solution. Brahmagupta also solved quadratic indeterminate equations of the form $ax^2 + c = y^2$ and $ax^2 - c = y^2$. For example, he solved $8x^2 + 1 = y^2$ and obtained the solutions $(x, y) = (1,3), (6,17), (35,99), (204,577), (1189,3363), \ldots$ .Later, Bhāskara (1114 – 1185) described the complete solution of Pell's

equation and developed a cyclic algorithm to produce a solution of equation $x^2 - Dy^2 = 1$ Bhāskara worked extensively with the more general equation $x^2 - Dy^2 = N$, and even solved the equation $Dx^2 + 1 = y^2$ for D=8, 11, 32, 61 and 67. When D = 61 he found $x = 226153980$, $y = 1776319049$. When D = 67 he found $x = 5967$, $y = 48842$.

Some applications of Bhāskara's equation include generation of the Pell sequence (with $D = 2$, N=1), arithmetic triangles (with $D = 3$) and generation of the Fibonacci sequence (with $D = 5$, N=1). It is known that all solutions of the equations $x^2 - 5y^2 = \pm 4$ are given by $y = \pm F_{2n+1}, Y = \pm L_{2n+1}; Y = \pm F_{2n}$ respectively, where F_n and L_n are respectively n^{th} Fibonacci number and Lucas number. Narayana Pandit (1340 – 1400) also found general solutions to Pell's equation and other quadratic indeterminate equations.

The topic interested the European mathematicians after a challenge given in 1657 by renowned mathematician Pierre de Fermat (1601 – 1665). He inspired some of his contemporaries to do the same. John Wallis, Frénicle, William Brouncker, Leonhard Euler, Joseph Louis Lagrange and others also contributed in the development of the theory of Pell's equations.

John Pell (1611 – 1685) was an English mathematician and an algebraist, but he hardly did any work with the equation $x^2 - Dy^2 = N$ that bears his name. In fact, Leonhard Euler credited the equation to Pell in a letter to Goldbach in 1730, and the name stuck. Lagrange in 1768 used the technique of Continued fraction to find the good approximations of $\sqrt{D}$ for the case N=1

Even today, research involving this equation continues to be very active and hundreds of articles dealing with this equation in various contexts have appeared in last some years. One of the main reasons for this interest is that the equation has a habit of popping up at an

unexpected place. It is also of great importance in solving the general second-degree Diophantine equation in two unknowns: $ax^2 + bxy + cy^2 + dx + ey + f = 0$ Furthermore, the problem of solving this equation is connected to that of determining the regulator, an important quantity of a real quadratic number field, and to solving the discrete logarithm problem in such structures. Today, this second problem is of interest to cryptographers.

Necessities for solving the Pell's equations:

We once again consider the Pell's equation

$$x^2 - Dy^2 = N \quad \text{.....(1)}$$

Suppose that (1) is solvable, and let $x = u, y = v$ be the integers satisfying (1). Then we call $u + v\sqrt{D}$ to be the solution of (1). We also denote this solution by (u, v) It will be called positive solution if both $u > 0$ and $v > 0$ We call a solution (u, v) of (1) to be trivial if $uv = 0$. Every nontrivial solution can be made into a positive solution by changing the sign of x *or* y.

Note: Here we note that by this we do not mean that $u + v\sqrt{D}$ is an integer. It is just another way of representing the solution (u, v) of (1).

If $x + y\sqrt{D}$ is the smallest positive solution of Pell's equation

$$x^2 - Dy^2 = 1 \quad \text{..... (2)}$$

then we call it as *basic solution* of (2). This equation is always satisfied trivially by $x = \pm 1, y = 0$. We thus call $(x, y) = (\pm 1, 0)$ to be trivial solutions.

Solving Pell's equation $x^2 - Dy^2 = 1$:

To find a nontrivial solution of $x^2 - Dy^2 = 1$ by elementary methods, rewrite the equation as $x^2 - Dy^2 + 1$: and then set y 1,2,3, ... until you reach a value where $Dy^2 + 1$ is a perfect square. Call that value and then we have a solution (x, y).

Example 1: Two positive solutions of $x^2 - 2y^2 = 1$ are (3,2) and (17,12), since $2y^2 + 1$ is a square when y = 2 and 12, where it has values $9 = 3^2$ and $289 = 7^2$. See below:

y	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$2y^2 + 1$	3	9	19	33											

Example 2: Three positive solutions of $x^2 - 3y^2 = 1$ are (2,1),(7,4) and (26,15), as shown by the table below

The following table gives the list of basic solutions (x, y) of the equation $x^2 - Dy^2 = 1$ for the non-square values of D for $2 \leq D \leq 24$.

D	2	3	5	6	7	8	10	11	12
x	3	2	9	5	8	3	19	10	7
y	2	1	4	2	3	1	6	3	2

D	13	14	15	17	18	19	20	21	22	23	24
x	649	15	4	33	17	170	9	55	197	24	5
y	180	4	1	8	4	39	2	12	42	5	1

The following theorem by *Lagrange* states that the Pell's equation $x^2 - Dy^2 = N$ always has solution if N = 1 It also gives the explicit formula for all the positive solutions.

Theorem 1: The Pell's equation

$$x^2 - Dy^2 = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

always has infinitely many positive solutions. If (x_1, y_1) is the basic solution (least positive solution) of (2) then all the positive solutions of (2) are given by (x_n, y_n) where x_n and y_n (are the integers determined from

$$x_n + y_n \sqrt{D} = (x_1 + y_1 \sqrt{D})^2; n = 1, 2, 3, \ldots..$$

Remark: Here (x_n, y_n) can be calculated algebraically from $x_n + y_n \sqrt{D} = (x_1 + y_1 \sqrt{D})^n$ expanding the left side, equating

coefficients of $\sqrt{D}$ on both sides, and equating the other terms on both sides.

This theorem now guarantees that when we tabulate $Dy^2 + 1$, it will always have a square value at some instance. We are guaranteed this search will eventually terminate, but we are not assured how long it will take. In fact, the smallest positive solution of $x^2 - Dy^2 = 1$ can be unusually large compared to the size of D. The table above illustrates this if we compare the smallest positive solutions when D = 12, 13, and 14. As more extreme examples, see in the table below the smallest positive solutions to $x^2 - 2y^2 = 1$ when D = 61 and 109 compared with nearby values of D

D	**60**	**61**	**62**		**108**	**109**	**110**
x	31	1766319049	63		1351	158070671986249	21
y	4	226153980	8		130	15140424455100	2

While Lagrange was the first person to give a proof of Theorem 1, in 1768, a century earlier Fermat claimed to have a proof and challenged other mathematicians in Europe to prove it too. In a letter in 1657 he wrote that one should try to find a positive solution to $x^2 - 61y^2 = 1$ and $x^2 - 109y^2 = 1$ Fermat had no idea that a nontrivial solution to $x^2 - 61y^2 = 1$ had been found in India (by Bhaskara II) 500 years before him.

About the solutions of Pell's equation $x^2 - Dy^2 = N$ Consider $u + v\sqrt{D}$ to be any solution of (1) and let $x + y\sqrt{D}$ is the basic solution of (2). Then it is easy to observe that

$$u + v\sqrt{D}\ X(x + y\sqrt{D})^{\pm 1} = (ux \pm Dvy) + (uy \pm vx)\sqrt{D}.$$

$$\begin{aligned}(ux \pm Dvy)^2 - D(uy \pm vx)^2 &= u^2x^2 + D^2v^2y^2 - Du^2y^2 - Dv^2x^2 \\ &= u^2(x^2 - Dy^2)^2 - Dv^2(x^2 - Dy^2) \\ &= (u^2 - Dv^2)(x^2 - Dy^2) = N\end{aligned}$$

Thus, the product $(u + v\sqrt{D}\ X\ (x + y\sqrt{D})^{\pm 1}$ is also a solution of (1). Suppose this new solution is $u' + v'\sqrt{D} = (u + v\sqrt{D})X(x + y\sqrt{D})^{\pm 1}$. This means that any solution of (1) when multiplied or divided by $x + y\sqrt{D}$ again gives the solution of (1). We call this new solution $u' + v'\sqrt{D}$ to be **associated** with the earlier solution $u + v\sqrt{D}$. The following gives the exact definition of this associativity. ***Definition:*** Two positive solutions $u_\alpha + v_\alpha\sqrt{D}$ and $u_\beta + v_\beta\sqrt{D}$ are said to be **associated** if there exists an integer t ` such that $u_\beta + v_\beta\sqrt{D} = (u_\beta + v_\beta\sqrt{D})(x + y\sqrt{D})^t; t = 0, \pm 1, 2$ ..... (3) If $u_\alpha + v_\alpha\sqrt{D}$ is any fixed solution of (1), then all the positive solutions given by (3) are said to be **associated with each other**. In this case, the set of all solutions associated with other forms a **class of solutions** of (1). For some solutions $u_\alpha + v_\alpha\sqrt{D}$ and $u_\beta + v_\beta\sqrt{D}$ of (1), if there does not exist t satisfying (3), then these two solutions are in different class of solutions of (1).

Fundamental solutions:

There is a quick way to generate other solutions from a single solution of Pell's equation. Before discussing this, we introduce the *fundamental solution* of the equation, $x^2 - Dy^2 = N$, from which all other positive integer solutions may be obtained.

Let **K** be any specific class consisting of all the solutions $u_i + v_i\sqrt{D}; i = 1,2,3, \ldots$. Which are associated with each other. Among all the solutions $u_i + v_i\sqrt{D}$; in a given class **K** we now choose a solution $u_1 + v_1\sqrt{D}$ in the following way: Let v_1 be the least non-negative value of v whichoccurs in **K**. In this case, u_1 is also uniquely determined. The solution $u_1 + v_1\sqrt{D}$ defined in this way is said to be *fundamental solution* of the class.

Thus, for the fundamental solution $u_1 + v_1\sqrt{D} \in K$, the number $|u_1|$ has the least value which is possible for $|u_1|$. In this case v_1 is also the least non-negative value of v_i which occurs in **K**. In other words, the fundamental solution is the smallest positive solution belonging to that class.

Since the fundamental solution $u_1 + v_1\sqrt{D}$ is the smallest positive solution of the class **K**, it can be observed that $u_1 + v_1\sqrt{D}$ is a fundamental solution of class **K** if and only if $u_1 + v_1\sqrt{D}\, X\, (x1 - y1\sqrt{D})$ is not a positive solution of (2). Thus if $u_\alpha + v_\alpha\sqrt{D}$ is any fixed fundamental solution then all the positive solutions obtained by (3) are associated with each other and all of them belong to some common class of solutions.

Moreover, we observe that if (1) is solvable, then it has only finite number of classes of solutions (and so finite number of fundamental solutions). We assume that (1) has exactly β classes of solutions. If $u_i + v_i\sqrt{D}; 1 \leq i \leq \beta$ are all the fundamental solutions of (1), then we write this as $u_i + v_i\sqrt{D}$ runs through all the fundamental solutions of (1)'.

Remark: If $u_1 + v_1\sqrt{D}$ is the smallest of all the fundamental solutions of (2) then for all its fundamental solutions $u_i + v_i\sqrt{D}$, , wehave

$u_i + v_i\sqrt{D} \leq u_1 + v_1\sqrt{D}\, X\, x_1 + y_1\sqrt{D}$; for all **i**.

This gives the upper bound for the fundamental solutions of (1).

The following theorem gives the bounds for the values of u, v occurring in the fundamental solutions.

Theorem 2: If $x + y\sqrt{D}$ is the basic solution of (2) and let $u + v\sqrt{D}$ be any fundamental solution of (1), then

$0 < |u| \leq \frac{1}{2}\sqrt{\frac{[x+1]|N|}{2}}$ and $0 \leq v \leq y\sqrt{\frac{|N|}{2(x+1)}}$.

Although the technique used above (and in the examples which will follow in next section) to find the fundamental solutions provides a general method to show that (2) has no solutions. Sometimes the equation has no solution can be proved simply using congruences. For instance, if we consider the Pell's equation $x^2 - 3y^2 = 2$, then it can be observed that this equation is not solvable. This is because from this equation, we have $x^2 \equiv 2 \pmod 3$ which has no solution.

But congruence methods do not always help to prove that any given Pell's equation has no solutions! The equation $x^2 - 37y^2 = 11$ considered in example 8 is one an example. We will prove that it has no solution, but it can be checked that the congruence $x^2 \equiv 11 \pmod{37}$has a solution14!

The following theorem states that (1) always has solution if N is a perfect square.

Theorem 3: For any positive, non-square integers D and any non-zero integer k, the Pell's equation $x^2 - Dy^2 = k^2$ always has a solution in integers.

At present there is no method to predict whether (1) is solvable or not for any given D and N. Although, some algorithms are available in literature which helps in hunt for the solutions of (1) for any given D and N.

Examples of fundamental solutions:

We now present some examples which will help to understand the idea of fundamental solutions more precisely.

Example3: Consider the Pell's equation $x^2 - 6y^2 = 3$.
The smallest positive solution of this equation is $(u, v) = (3,1)$. Also, the equation $x^2 - 6y^2 = 1$. has basic solution $(x, y) = (5,2)$. Then by theorem 2, any fundamental solution $u + v\sqrt{D}$ has bounds

$$0 < |u| \leq \frac{1}{2}\sqrt{\frac{6 \times 3}{2}} = 1.5 \text{ and } 0 \leq v \leq 2\sqrt{\frac{3}{2 \times 6}} = 1.$$

Thus, the solution $(3,1)$ is the only fundamental solution of $x^2 - 6y^2 = 3$. Here we also observe that any fundamental solution must occur before the number $3 + \sqrt{6}$ $(5 + 2\sqrt{6} = 27 + 11\sqrt{6}$ that is before the solution $(27,11)$. Now $(u + v\sqrt{6})$ $(x - y\sqrt{6} = (3+\sqrt{6}\left(5 - 2\sqrt{6}\right) = 3 - \sqrt{6}$, which gives negative solution $3, -1$ of the given equation. Thus $3 + \sqrt{6}$ is the fundamental solution.

Example 4: Consider the Pell's equation $x^2 - 7y^2 = 57$.

The smallest positive solution of this equation is $(u, v) = (8,1)$. Also, the equation $x^2 - 7y^2 = 1$ has basic solution $(x, y) = (8,3)$.. Then any fundamental solution $u + v\sqrt{D}$ has bounds

$0 < |u| \leq \frac{1}{2}\sqrt{\frac{9 \times 57}{2}} \approx 8.01$ and $0 \leq v \leq 3\sqrt{\frac{57}{2x9}} = 5.34$. It can be now easily observed that the equation $x^2 - 7y^2 = 57$ are satisfied by only $(u, v) = (8,1), (13,4)$ in the above range. Thus, these are the two fundamental solutions.

We can even verify that $(u + v\sqrt{7})(x - y\sqrt{7}) = (8 + \sqrt{7})(8 - 3\sqrt{7}) - 43 - 16\sqrt{7}$, which is negative. Also, if we consider $(u, v) = (13,4)$, then

$$(u + v\sqrt{7})(x - y\sqrt{7}) = (13 + 4\sqrt{7})(8 - 3\sqrt{7}) - 20 - 7\sqrt{7},$$

a negative quantity. This too justifies that (8,1) and (13,4) are the fundamental solutions of $x^2 - 7y^2 = 57$.

All the solutions of Pell's equation $x^2 - Dy^2 = N$:

As discussed above, Pell's equation (2) always has a solution. But the generalized Pell's equation(1) may or may not have solutions in integers. It is proved that if solvable, then (1) also always has infinite number of solutions in integers. The following result gives an explicit formula to find all the solutions of Pell's equation (1).

Theorem 4: If $u_i + v_i\sqrt{D}$ runs through all the fundamental solutions of (1) and if $x + y\sqrt{D}$ is the basic solution of (2), then all the positive solutions of (1) are given by

$$x_{i,n} + y_{i,n}\sqrt{D} = (u_i + v_i\sqrt{D})(x + y\sqrt{D})^2; = 0,1,2, \ldots..$$

Examples for the solutions of Pell's equation $x^2 - Dy^2 = N$:

Example 5: Consider the Pell's equation $x^2 - 3y^2 = 6$.

The smallest positive solution of this equation is $(u, v) = (3,1)$. Also, clearly the equation $x^2 - 3y^2 = 1$ has basic solution $(x, y) = (2,1)$. Using theorem 2, it can be observed that $x^2 - 3y^2 = 6$ has only one fundamental solution (3,1). Then above theorem states that all the solutions of $x^2 - 3y^2 = 6$ are given by

$$x_{i,n} + y_{i,n}\sqrt{3} = (3 + \sqrt{3})(2 + \sqrt{3})^n; n = 0,1,2, \ldots.$$

This now gives the infinite number of solutions of $x^2 - 3y^2 = 6$ as $(3,1), (9,5), (33, 19), \ldots..$.

Example 6: Consider the Pell's equation $x^2 - 19y^2 = 36$.

The smallest positive solution of this equation is $(u, v) = (6,0)$. Also, the equation $x^2 - 19y^2 = 1$. has basic solution $(x, y) = (170,39)$. If $u + v\sqrt{D}$ runs through the fundamental solutions of $x^2 - 19y^2 = 36$, , then $0 \leq v \leq 39\sqrt{\frac{36}{2x171}} \approx 12.65$. Then it can be verified that (6,0) and (44,10) are the two fundamental solutions.

Thus, all the solutions of $\boldsymbol{x^2 - 194y^2 = 36}$ are given by two classes of solutions

$$x_{i,n} + y_{i,n}\sqrt{19} = 6(170 + 39\sqrt{19})^n \text{ and}$$
$$x_{i,n} + y_{i,n}\sqrt{19} = (44 + 10\sqrt{19})(170 + 39\sqrt{19})^n; n = 0,1,2, \ldots$$

Example7: Consider the equation $\boldsymbol{x^2 - 194y^2 = -1}$.

Smallest solution of $\boldsymbol{x^2 - 194y^2 = 1}$. **is** $(x, y) = (195,14)$. If $u + v\sqrt{D}$ is any fundamental solution of $\boldsymbol{x^2 - 194y^2 = -1}$, then $0 \leq$ v $\leq 14\sqrt{\frac{1}{2x196}} \approx 0.71$. Thus, the only choice for the value of v is v = 0. But in that case we have $\boldsymbol{x^2}$**=-1.** Clearly this has no real (integer) solution. Hence the equation $\boldsymbol{x^2 - 194y^2 = -1}$ has no integer solution.

Example 8: Consider the equation $\boldsymbol{x^2 - 37y^2 = 11}$.

The basic solution of $\boldsymbol{x^2 - 37y^2 = 1}$ is $(x, y) = (73,12)$. If $u + v\sqrt{D}$ is any fundamental solution of $\boldsymbol{x^2 - 37y^2 = 11}$, then $0 \leq$ v $\leq 12\sqrt{\frac{11}{2x74}} \approx 6.58$. In given equation $\boldsymbol{x^2 - 37y^2 = 11}$, by taking the value of $v = y$ from *0* to *6*, it can be observed that the equation $\boldsymbol{x^2 - 37y^2 = 11}$ does not have any x satisfying it. Thus, the given equation has no solution.

NegativePell'sequation:

While the Pell's equation (2) always has infinitely many non-trivial solutions, the analogous equation, the **negative Pellian equation** $\boldsymbol{x^2 - Dy^2 = -1}$ is much more mysterious. The interesting point is that there are many non-square values of D for which this equation is not solvable. One well known limitation is $D \equiv 1, 2 \pmod 4$& with no 4n+3 factors,

but this is not sufficient. If $p \mid D$, then reducing modulo p tells that the congruence $x^2 \equiv -1 \pmod{p}$ must be solvable,
when $p \equiv 1 \pmod{4}$ or $p = 2$. In this case, by a well-known theorem of Fermat, it can be clearly seen that D must be sum of two squares.
However not all such D has the property that $x^2 - Dy^2 = -1$ is solvable: the smallest counterexample is D = 34. These has been summarized in the following theorem:
Theorem 5: If $x^2 - Dy^2 = -1$ is solvable, then D is not divisible by 4 and D does not have a prime factor congruent to 3 modulo 4.
I conclude this article by mentioning that any Pell's equation which is solvable, can even be solved by using the technique of continued fractions, which is a matter of some other article.

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Mathematics in Music

-Paraj Modi

[Paraj Modi is a third year medical student in Ahmedabad. She loves medicine, mathematics and art forms. She plays tabla and practices origami as a meditative art. This article is a revised version of her own article which she wrote for 'Suganitam' October 2023 issue published by Gujarat Ganit Mandal. Necessary formalities are done. –Editor].

Beautiful Math: Music

Paraj Modi

नृत्यं वाद्यानुगम प्रोक्तं गीतानुवती च। *(Sangeet Ratnakar 1|21)*

Vocals, instrumentals, and dance collectively constitute music.

India has a rich tradition of Classical Music or *Shastriya Sangeet*. This system of music is not just an invaluable heritage, but also a timeless testimony, unifying arts with sciences and mathematics.

We shall try to explore mathematics in various aspects of music, such as

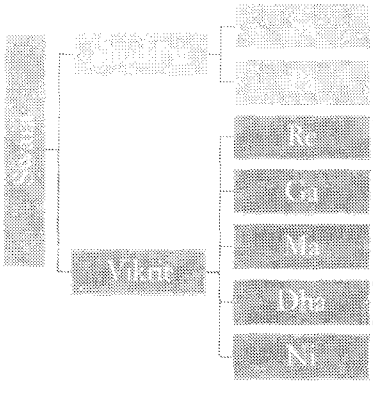
- *Swara* (vocals and stringed instruments)
- *Taal* (Percussion)
- *Nritya* (Dance)

1. Swara

Swara consists of various notes like *Sa, Re, Ga,* etc. that constitutes various ragas. *Swara(s)* can be classified into two basic streams: *Shuddh* and *Vikrit*. In a *saptak* (octave), there are 12 notes, including both the *shuddh* and *vikrit swara(s).*

Shuddha swara(s) do not have alternate frequencies and hence, are also called '*Achal (immovable) Swara(s)*'.

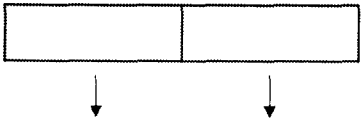
Sr. No.	Name of Swara	Indian Symbol	Western Symbol
1	Shadj	S / सा	C
2	Komal Rishabh	r / रे	C# / Db
3	Rishabh	R / रे	D
4	Komal Gandhar	g / ग	D# / Eb
5	Gandhar	G / ग	E
6	Madhyam	M / म	F
7	Madhyam Teevra	M# / म	F# / Gb
8	Pancham	P / प	G
9	Komal Dhaivat	d / ध	G# / Ab
10	Dhaivat	D / ध	A
11	Komal Nishaad	n / नि	A# / Bb
12	Nishaad	N / नि	B



Vikrit swara(s), on the other hand, may have alternate frequencies, either higher (*Teevra*) or lower (*Komal*) than the standard *swara*. Ragas can be classified into various families, better known as '*thaat*'(s), having similar *swara(s)*. In Hindustani Classical Music, there are 10 *thaat*(s). Mathematics is particularly useful in determining how many ragas can be created in of one *thaat*. The concept of permutation and combination functions here.

1.1.Classification of ragas

Ragas can be classified into three types based on their structure. The basic structure of a raga consists of *Aaroh* (ascending scale) and *Avroh* (descending scale). Depending on the number of notes in both *Aaroh* and *Avroh*, ragas can be classified into *Sampurna* (7 notes are included), *Shadav* (6 notes are included) and *Odav* (5 notes are included). These are known as the *Jaati*(s) of Ragas.

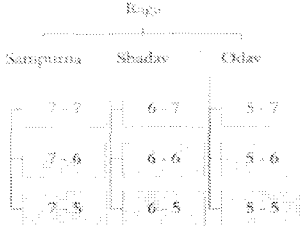


3 x 3 = 9

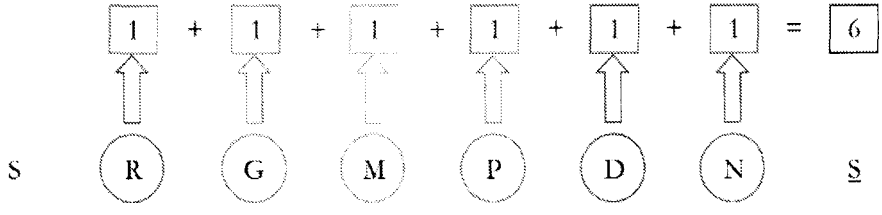
In order to determine how many *Jaati*(s) are possible, wecan run a simple permutation. Let us assume we have two vacant spots (*Aaroh* and *Avroh*). We can fill these up with 3 possibilities (*Aaroh* or *Avroh* can have aminimum of 5 *swara(s)* and maximum of 7 *swara(s)*, i.e., the possibilities are 5, 6, and 7 *swara(s)*). Thus, 9 *Jaati*(s) of ragas are possible.

1.2. How many ragas can be created using permutation combination?

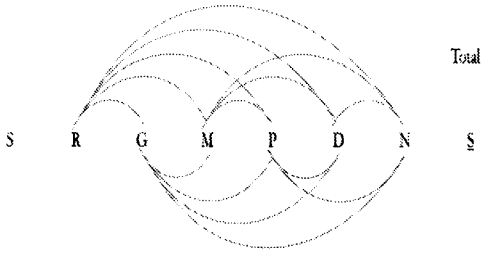
Sr. No.	Combination of Swaras	Number of Combinations
1	Sampurna – Sampurna (7 – 7)	1
2	Sampurna – Shadav (7 – 6)	6
3	Sampurna – Odav (7 – 5)	15
4	Shadav – Sampurna (6 – 7)	6
5	Shadav – Shadav (6 – 6)	36
6	Shadav – Odav (6 – 5)	90
7	Odav – Sampurna (5 – 7)	15
8	Odav – Shadav (5 – 6)	90
9	Odav – Odav (5 – 5)	225
Total Combinations		484



Different ragas can be created by combination of various *swara(s)*.Consider the situation of *Sampurna – Sampurna*. Here, there are 7 *swara(s)* in both *aaroh* and *avroh* respectively. Only 1 combination is possible, i.e., *Sa-Re-Ga-Ma-Pa-Dha-Ni* and *Ni-Dha-Pa-Ma-Ga-Re-Sa.*



Let us consider *Sampurna – Shadav* or *Shadav – Sampurna*. We have 7 *swara*(s) on one hand and 6 on the other. Hence, we can skip 1 *swara* each (except Sa) during the 6 *swara aaroh* (ascent) or descent (*avroh*) respectively. Hence, we have 6 combinations.



Similarly, in case of *Odav* combinations, either in *Aaroh* or *Avroh,* 2 notes can be skipped together either in ascending or descending pattern, as shown in the diagram. This leaves us with 15 combinations.

We can approach this problem a bit more mathematically as well, by using combinations. We can never drop *Sa* out of any combination of *swara(s)*. Hence, we have 6 notes to choose from and fill up 6 spaces in case of *Shadav* and 5 spaces in case of *Odav.*

$$nc_r = \frac{n!}{(n-r)!\,r!}$$

$$6c_1(Shadav) = \frac{6!}{(6-1)!} = 6$$

Here, we are taking r = 1 since we have to omit 1 *swara* in *Shadav* and n = 6 since we have 6 *swara(s)* to choose from.

	Aaroh Combinations Possible (x)	*Avroh* Combinations Possible (y)	Total Raga Combinations Possible (xy)
Sampurna – Sampurna	1	1	1
Sampurna – Shadav	1	6	6
Sampurna – Odav	1	15	15
Shadav – Sampurna	6	1	6
Shadav – Shadav	6	6	36
Shadav – Odav	6	15	90
Odav – Sampurna	15	1	15
Odav – Shadav	15	6	90
Odav – Odav	15	15	225
Total Combinations			484

$$6c_2(Odav) = \frac{6!}{(6-2)!\,2!} = 15$$

Similarly, for Odav, we have r = 2 since 2 notes have to be omitted.

The total number of combinations including both *aaroh* and *avroh* can be calculated by multiplying *aaroh* and *avroh* combinations possible.

Thus, a total of 484 ragas can be created in one *thaat*. Considering that there are 10 *thaat*(s), 4840 ragas can potentially exist.

This number helps us appreciate various permutations and combinations. However, alterations may be made considering that *Ma* and *Pa* cannot be omitted together.

1.3. Vaadi – Samvadi Swara(s)

Sa is the basic *swara* relative to which the entire *saptak* thrives. Let us consider the frequency of *Sa* to be 240 Hz in a particular scale. Mathematics comes in handy when we wish to know the frequency of one of the missing pairing *Vaadi - Samvadi Swara(s). Vaadi swara* is the prominent *swara* of a raga, which is supported by the *Samvadi swara.*

Samvadi Swara, by convention lies at a frequency 1.5 times higher than that of *Vaadi Swara.*

Hence, if the *Vaadi Swara* is *Sa*, *Samvadi Swara* would have a frequency of 240 Hz x 1.5 = 360 Hz, which is the frequency of *Pa*. Hence, *Sa* and *Pa* form a pair of *Vaadi – Samvadi Swara.*

Sa – Pa pair has a difference of 13 *shruti*(s) between them. *Sa – Pa* relationship is considered *shreshta* (supreme) and they are *parasparpurvak samvaadswar* (mutually complementary). They sound the most melodious as each others 'complements.

$\boldsymbol{V = 1.5 \times S}$ *(V = frequency of Vaadi Swara; S = frequency of Samvadi Swara)*

Similarly, *Sa* and *Ma* can also effectively form a *Vaadi – Samvadi* pair. However, in that case, the ratio of their frequencies would be 3:4. They have 9 *shruti*(s) between them.

$$\frac{\text{sa}}{\text{ma}} = \frac{240}{320} = \frac{3}{4}$$

Similarly, *Sa* - $\overline{sa}$*(Shadaj – Taarshadaj) samvaad* is known as *swayambhaav.*

$$\frac{\bar{sa}}{sa} = \frac{2}{1}$$

Frequency and length of String

Swara	Length of string (inches)	Frequency (Hz)
Sa	36	240
Re	32	270
Ga	30	288
Ma	27	320
Pa	24	360
Dha	21.33	405
Ni	20	430

The data consists of length of string of a *Veena* corresponding to their rough experimental frequencies. In order to find frequency of a *swara* using the length of its string, we make use of simple ratios. Length of string is indirectly proportional to frequency pertaining to a *swara*.

For example, we know that the frequency of *Sa* is 240 Hz, length of string of *Sa* is 36 inches and length of string of *Ma* is 27 inches. How do we find frequency of Ma?

We equate ratios of lengths of strings of various *swara(s)* inverse to their respective frequencies. Let frequency of *Ma* be F. Hence,

$$\frac{36}{27} = \frac{4}{3} = \frac{F}{240} \Rightarrow F = 320\ H_Z$$

$$F = \frac{36}{y}\ \chi 240$$

(F= freq. of any swara, Y =length of string

We can also perform back calculation and obtain length of string of a *swara* using its frequency. For example, frequency of *Sa* is 240 Hz and

frequency of *Ma* is 320 Hz. Length of string of *Sa* is 36 inches. Again we equate ratios of frequencies of *swara(s)* inverse to their length of strings.

Strings:

$$\frac{320}{240} = \frac{36}{\mathrm{L}} \Rightarrow L = 27\ inches.$$

A basic conclusion can be drawn here:

Pitch α F/ Length of string.

1.1.Frequencies of semitones

There exist 12 semitones in an octave, i.e., there are 12 basic *swara(s)* (including *shuddha* and *vikrit*) in a *saptak*. These differ in frequency by a factor of $2^{1/12}$ = 1.0595 times the frequency of the *swara* preceding it.

Sr. No.	Swara	Ratio of Frequency w.r.t. S
1	S	1
2	r	16/15
3	R	10/9
4	g	6/5
5	G	5/4
6	M	4/3
7	M#	45/32
8	P	3/2
9	d	8/5
10	D	5/3
11	n	9/5
12	N	15/8
13	S (of the next octave)	2

$$f_n = f_b \, x \, 2^{X/12}$$

Also, the frequencies of all *swara(s)* of a *saptak* have been determined with respect to their ratios with the *Sa* of the octave. It is given in the following table.What is interesting is the ratio of a particular note in the preceding (*mandra saptak*), existing (*madhya saptak*) and succeeding (*taar saptak*) octave would be 1:2:3.

1.1.Merukhand

Number of *swaras* = n	Number of unique combinations possible = n!	*Swaras* considered (here)	Combinations possible
3	3! = 3x2x1 = 6	S	SRG
			SGR
		R	RSG
			RGS
		G	GSR
			GRS

Meru means '*axis* and '*khand* means '*parts*. It is a traditional practice wherein different *swara(s)* are arranged in different patterns. The number of patterns are the permutations possible with the given number of *swara(s).* For example, if we take two swara(s) – Sa and Re. The number of swara(s) is 2. 2! = 2 Hence the number of ways it can be sung is 'Sa-Re' and 'Re-Sa'. Similarly, when we take 3 notes, number of permutations possible are 3! = 3x2x1 = 6.

2. *Taal*

Indian Classical Music has rhythm as an integral part. These instruments are known as percussion instruments or *Taal Vaadhya*. Few of these include *Mridangam, Tabla, Dholak, Pakhawaj, Ghatam*, etc.

Let us consider the use of mathematics in *Tabla*. *Tabla* compositions

form patterns within themselves. Even today, *Tabla* compositions tend to be delivered orally from a Guru to a *Shishya*. They are counted on fingers, spoken rhythmically with hand movements (*padhant*) and memorized.

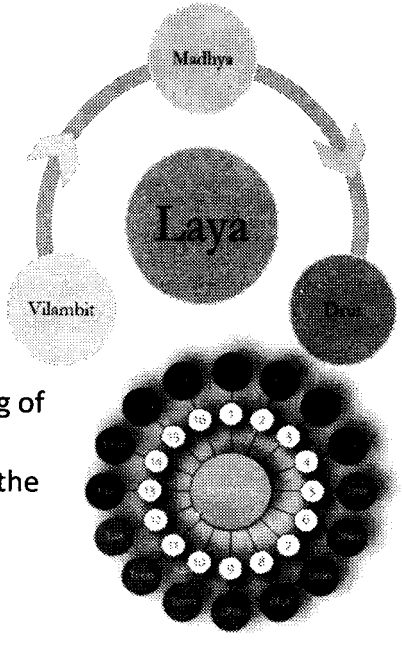
2.1.Laya

Laya refers to the speed or tempo of a composition, it forms a backbone of any rendition in music. Mathematics constitutes the concept of *laya*. There are three fundamental types of *laya*: *vilambit* (slow), *Madhya* (medium), *Drut* (fast).

These can be visualized using a say '*Lay-ometer*' like an Odometer.

2.2.Taal Chakra

A *taal* refers to a cycle of fixed number of beats: '*maatraa*'(s), say 'n'. A *taal* has the following basic components:



Maatraa: number of beats
Khand: Divisions in a *taal* consisting of sets of *maatraa*(s)

- *Bol*: Notations that constitute the *maatraa*(s)
- Most common *taal*(s) include:
- *Teentaal* (16 beats)
- *Kehervaa* (8 beats)
- *Dadra* (6 beats)

- *Ektaal* (12 beats)
- *Jhaptaal* (10 beats)
- *Roopak* (7 beats)

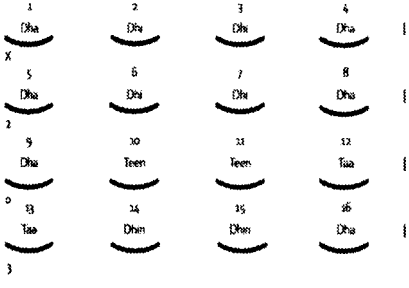
A visual representation of *Teentaal* is shown. Here four differentcolors code four *khand*(s) in which the *taal* is divided.
Let us consider playing a *taal*, say *Teentaal* in four different *laya*(s).*Maatraa*: number of beats

- *Khand*: Divisions in a *taal* consisting of sets of *maatraa*(s)
- *Bol*: Notations that constitute the *maatraa*(s) Most common *taal*(s) include:

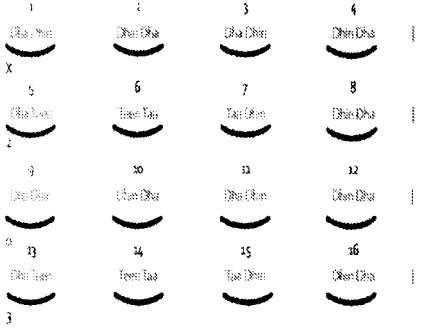
- *Teentaal* (16 beats)
- *Kehervaa* (8 beats)
- *Dadra* (6 beats)
- *Ektaal* (12 beats)
- *Jhaptaal* (10 beats)
- *Roopak* (7 beats)

A visual representation of *Teentaal* is shown. Here four differentcolors code four *khand*(s) in which the *taal* is divided. Let us consider playing a *taal*, say *Teentaal* in four different *laya*(s).

While playing a composition, we try to accommodate *'n'* number of beats in one beat. Thus, the composition has to be played *'n'* times to arrive at the first beat back again.

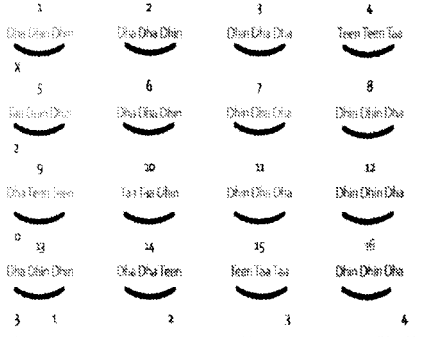


Single Beat: The original *taal* is played once. Each cycle (*Aavartan*) takes 16 beats.



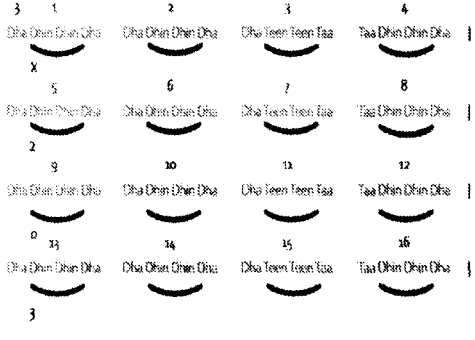
Double Beat (Dugun): The *taal* is played twice since two beats are accommodated in one beat. Color coding shows the *taal* played twice.

Each cycle (*Aavartan*) takes 8 beats.



Triple Beat (Trigun): The *taal* is played thrice, since three beats of the *taal* collectively occupy one beat.

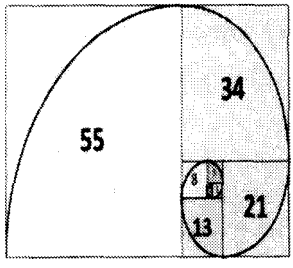
Each cycle (*Aavartan*) takes roughly 5.3 beats



Quad Beat (Chaugun): The *taal* is played four times, since four beats collectively occupy one beat.

Each cycle (*Aavartan*) takes 4 beats.

2.3. The *Hemachandra* – Fibonacci Series



The *Hemachandra* numbers or Fibonacci series consists of numbers in which the successive terms are the sums of the preceding two consecutive terms. It is as follows:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610....

These are also visually represented as the Golden Ratio.

Let us consider two sets of *bol*(s) (notations) in *Tabla*, namely, *'Dha Tee'* and *'Na'*. The first one is a long syllable and the second one is a short syllable. Let us try to fill 8 beats using various combinations of these two *bol*(s).

Let the number of beats be 'n' = 8

$$H_n = H_{n-1} + H_{n-2}$$

$$H_n \; refers \; n^{th} \; number \; in \; the \; Fibonacci \; Series$$

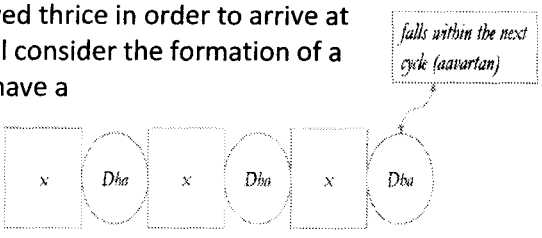
$$H_8 = H_7 + H_6$$

$$H_8 = 21 + 13 \; (Refer \; the \; above \; Fibonacci \; sequence)$$

Thus, 34 combinations can be made by different combinations of these sets of notes. This method is used in elaborate compositions like *Kaayadaa*(s) in *Tabla* and ancient Sanskrit poetry.

1.1.Concept of *Tihaai*

Tihaai refers to a set of *bol*(s) played thrice in order to arrive at the *sam* (first beat). Here, we shall consider the formation of a simple *tihaai*. A *tihaai* can either have a pause between the set to bol(s) to be played thrice. Such *tihaai*(s) are called *dumdaar tihaai*(s). On the other hand, certain *tihaai* may not have any gap and the *bol*(s) are played successively without any pause. Such *tihaai*(s) are known as *bedum tihaai*(s).



Let us try to form a *tihaai* in *Teentaal* (16 beats). Say we take a gap of 2 beat between the *bol*(s) to be played thrice successively (hence a gap of 2 beats X 2 = 4 beats).

$$16 - 4 = 12$$

Since the *bol*(s) will be played thrice, we need to divide the remaining beats by 3.

$$12/3 = 4$$

Hence, the *bol*(s) should accommodate within 4 beats.

On the other hand, say we have a fixed set of *bol*(s) that occupy 10 beats, we multiply them by 3.

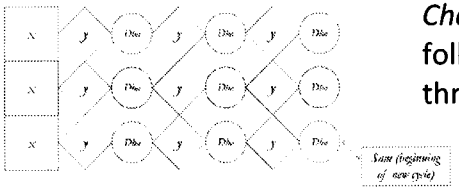
$$10 \times 3 = 30$$

The closest multiple of 16 (since we are forming a *tihaai* in *Teentaal* which has 16 beats) to 30 is 32. Hence, the *tihaai* will take 2 cycles of the *taal* (2 *aavartan*(s)) to complete.

Accordingly, we have 2 empty beats (32 – 30 = 2). Since we have to take the gap twice between the three times we play the *bol*(s), each gap would account for 1 beat.

This way we can form a *tihaai* from either the number of beats or pauses of our choice.

2.4. Chakradaar



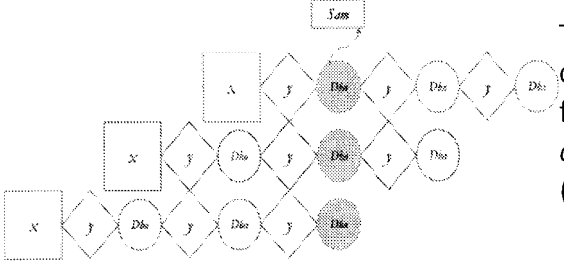
***Chakradaar* refers to a set of *bol*(s) followed by a *tihaai*, and is played thrice in order to arrive at the *sam*.**

Chakradaar can be formed in a similar fashion as the *tihaai*. It would depend on the number of beats in *bol*(s), the number of beats in pause, and number of beats in *tihaai*.

Chakradaar can further be of 2 major special types, namely *Farmayashi* and *Kamaali Chakradaar*.

2.6. Farmayashi Chakradaar

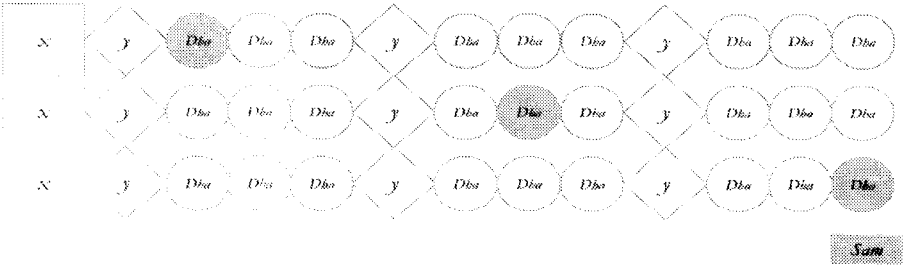
Farmayashi Chakradaar is a special kind of *Chakradaar*, wherein the 1st *Dha* of 1st *tihaai*, 2nd *Dha* of 2nd *tihaai* and 3rd *Dha* of 3rd *tihaai* arrive on the *sam*.



The following table shows details about the formation of a *farmayashi chakradaar* in *Teentaal* (16 beats).

The beat from which we wish to begin the Parmyoshi Chakradaar	*x = beats occupied by the bol(s)*	*y + Dha = mukhda (notes of tihaai including Dha)*	*Pause / gap to be taken between the composition (played thrice)*
1	12	5	0
2	11	5	1
3	4	11	0
4	8	6	0
5	12	1	0
6	16	12	0
7	4	7	0
8	8	2	0
9	12	13	0
10	16	8	0
11	4	3	0
12	8	14	0
13	12	9	0
14	16	4	0
15	4	15	0
16	8	10	0

2.7.Kamaali Chakradaar



Kamaali Chakradaar is yet another unique *chakradaar* wherein 3 *Dha*(s) are played after each *mukhda* (notations of *tihaai*). Hence, there are in total 9 *Dha*(s) in each cycle. Since there are 3 cycles to be played in order to finally arrive at the *sam*, there are total 27 *Dha*(s). The unique feature is that the 1st *Dha* of the 1st *mukhda*, 2nd *Dha* of the 2nd *mukhda* and 3rd *Dha* of the 3rd *mukhda* arrive at the *sam*.

The following table shows the details about making a kamaali chakradaar in Teentaal (16 beats).

The beat from which we wish to begin the Kamnasi Chakraduar	*x = beats occupied by the bol(s)*	*y = matkhdis (notes of thaat)*	*Number of Dhaa(s)*	*Pause / gap to be taken between the composition (played thrice)*
1	10	6	3	1
2	9	6	3	2
3	7	7	3	0
4	6	7	3	1
5	5	7	3	2
6	3	10	3	0
7	2	8	3	1
8	1	8	3	2
9	15	6	3	0
10	14	6	3	1
11	13	6	3	2
12	11	10	3	0
13	10	10	3	1
14	9	10	3	2
15	7	11	3	0
16	6	11	3	1

Conclusion and personal remarks

Indian classical music is one of the most beautiful creations of God. It's profundity that reflects through its simplicity, renders me in awe of how the *sur*, *taal* and *nritya* accurately depict myriads of human emotions. While Indian classical music prompts us to embrace these emotions, it also helps us rise above them. It liberates us and guides us towards a higher being. For me, listening to classical music or performing is a blissful and deeply ecstatic experience.

Mathematics, on the other hand, is a marvelous language that communicates the nuances of the mechanisms of the universe. Indian classical music subtly employs various mathematical concepts and depicts them as phenomenal exhibits of art.

This union of two of my most favorite subjects intrigued me and prompted me to write this article. The motive is not only to fascinate the readers with the marvel of our roots, but also to encourage my generation and the upcoming ones to appreciate the depth of Indian classical music, both artistically and scientifically.

I encourage all those who read this article, to delve into the realms of Indian classical music. I assure you it will embrace you with its peace, warmth and love.

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  - *[Link: https://www.urmimusic.com/post/merukhand]*
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  - *[Link: https://www.youtube.com/watch?v=siFBqH-LaQQ]*

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Ellipse, Circle, Ellipse, Circle .....

Meghraj Bhatt

We know that ellipse is one of the conic sections. Its algebraic equation is

$$\frac{x^2}{a^3} + \frac{y^2}{b^2} = 1 \quad \text{.....(1)}$$

When a > b; geometrically it is represented by a curve shown in fig. (1)

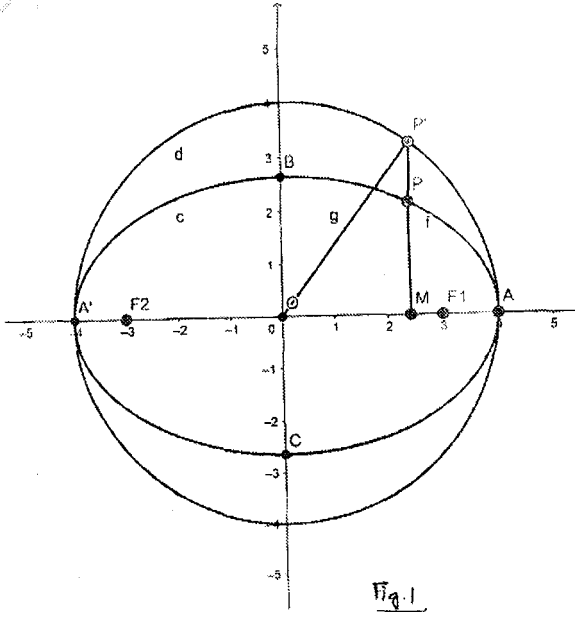


Fig. 1

AA' = 2a is called the major axis and BB'=2b is called the minor axis. A Circle with origin as center and Radius = a is shown which is called the auxiliary circle for this elipse.

$P(a\cos\theta, a\sin\theta)$. $\angle P'OA = \theta$ and is called the eccentric angle of the point P.

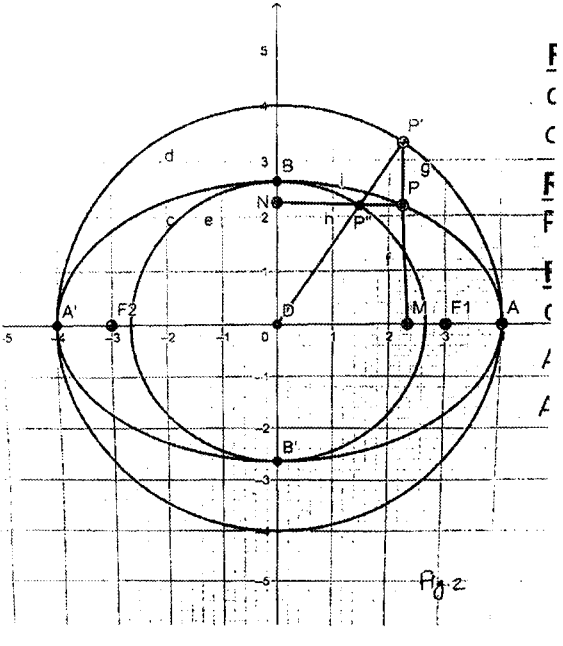
If we draw a circle with center at origin and radius = b, we get a circle inside the ellipse.

We will call it the Sub-auxiliary circle. Let us draw PN ⊥ Y –axis which Intersects the inner circle _in P" Where P" ($b\cos\theta, b\sin\theta$_) using The slope of OP" and p"P',it is easy to prove that O,P" and P' are Colinear. Hence,

Result -1 : Eccentric angle can be defined by that sub-auxiliary circle also,

as shown.

Result -2 : P"P = (a-b) $\cos\theta$: $PP' = (a-b)\sin\theta_{and} P'P'' = (a-b)$.



Result – 3: Area of the sub auxiliary circle = $A_1 = \pi b^2$

Area of the ellipse = $A_2 = \pi ab$

Area of the auxiliary circle = $A_3 = \pi a^2$

Hence, $\frac{A_2}{A_1} = \frac{A_3}{A_2} = \frac{A_3 - A_2}{A_2 - A_1} = \frac{a}{b}$ where $A_3 - A_2$ is the area of the auxiliary circle without ellipse and $A_2 - A_1$ is the area of the ellipse without the sub-auxiliary circle.

The ellipse with the eccentricity $e = \frac{1}{\sqrt{2}}$, we call it a Special Ellipse

Result -4: In the special ellipse, the sub-auxiliary circle will pass through its focus.

Result -5 : In the special ellipse, the eccentric angle of the latus rectum in the 1st quad. is $\frac{\pi}{4}$.

Result – 6: Let the ellipse given by eq. (1) earlier be a special ellipse. i.e.

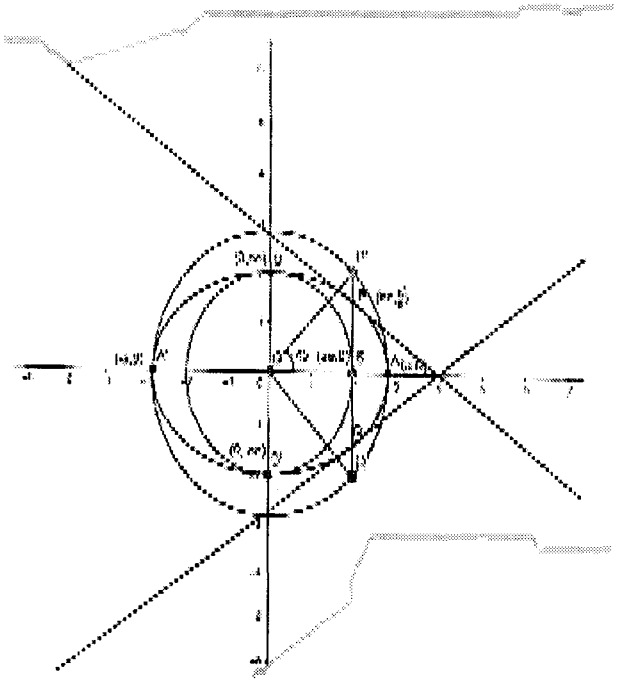
$$\frac{x^2}{a^3} + \frac{y^2}{b^2} = 1, a > b, e = \frac{1}{\sqrt{2}}.$$

$\therefore a = \sqrt{2}\ b$ ..... (2)

The equation of the auxiliary circle is $x^2 + y^2 = a^2$ .....(3)

The equation of the sub-auxiliary circle is $x^2 + y^2 = b^2$ .....(4)

Suppose the line y=ma + c touches the sub – auxiliary circle. Then it is easy to show That $c = b\sqrt{1 + m^2}$



$\therefore y = mx + b\sqrt{1+m^2}$ is the equation of the tangent with slope m to the sub-auxiliary circle. if this tangent is passing through $Q(x_1, y_1)$, then

$Y_1 - mx_1 = b\sqrt{1+m^2} \Rightarrow (x_1^2 - b^2)m^2 - 2x_1y_1m + (y_1^2 - b^2)$ .....(5)

Which gives two values m_1 *and* m_2 . If they are mutually perpendicular, then

$m_1m_2 = -1 \Rightarrow \frac{y_1^2 - b^2}{x_1^2 - b^2} = -1 \Rightarrow y_1^2 + y_1^2 = 2b^2 = \left(\sqrt{2}\, b\right)^2 = a^2[by(2)]$ .....(6)

Which is the equation of the locus of Q but (6) is the equation of the auxiliary circle!

Hence the result: In a special ellipse, the locus of the mutually per pendicular Tangents to the sub auxiliary circle is the auxiliary circle.

Now let us have some notations:

$C_1: x^2 + y^2 = b^2$ $(sub - auxiliary\ circle)$

$C_2: x^2 + y^2 = a^2$ (auxiliary circle)

$E_1: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b, e = 1/\sqrt{2}.$

Now if we consider two mutually perpendicular tangents of C_2, it will be a circle

$C_3: x^2 + y^2 = 2a^2 = (\sqrt{2}\,a)^2$ *which is a circle with radius* $\sqrt{2}\,a$ *i.e.*

$a = \frac{1}{\sqrt{2}}c = ec$ *which will be a new ellipse with equation*

$E_{23}: \frac{x^2}{c^2} + \frac{y^2}{a^2} = 1$ ..... (7)

Now for this ellipse E_{23}, C_2 will be the sub-auxiliary circle and C_3 will be the auxiliary circle. Again, if we consider two mutually perpendicular tangents of C_3, we get a circle $C_4: x^2 + y^2 = 2c^2 = (\sqrt{2}\,c)^2$ with radius d = $\sqrt{2}\,c$. Again we get a new special ellipse between C_3 and C_4 i.e. C_{34}

with equation : $\frac{x^2}{d^2} + \frac{y^2}{c^2} = 1$. In this way we get a series of concentric ellipses and circles and hence my title !

Mathematics News

By: Dr. Devbhadra V. Shah, Associate Professor, Dept. of Mathematics, Veer Narmad South Gujarat University, Surat, Gujarat, dvshah@vnsgu.ac.in

In their best performance, India bags 4th place in International Mathematical Olympiad 2024

A six - member team representing India has secured six medals – four Gold, one Silver and one Honourable mention – at the 65th International Mathematical Olympiad (IMO 2024) held at Bath, UK during 11 – 22 July, 2024. The team finished fourth among the 108 countries that participated in the competition.

Adhitya Mangudy Venkata Ganesh (Pune) (35 points) *Ananda Bhaduri* (Guwahati) (32 points), *Kanav Talwar* (Noida) (31 points) and *Rushil Mathur* (Mumbai) (30 points) won the gold medal. *Arjun Gupta* (Delhi) (24 points) won the silver medal, while *Siddharth Chopra* (Pune) (15 points) won the honourable mention.

Team USA, China, and South Korea finished as the top three winners. At fourth place, India's total score at the Olympiad is 167 just one mark behind South Korea's third spot. This is the best performance by an Indian in IMO both in terms of number of Gold medals won and rank achieved. India's previous best rank achieved until 2024 was rank 7 at IMO 1998 and IMO 2001.

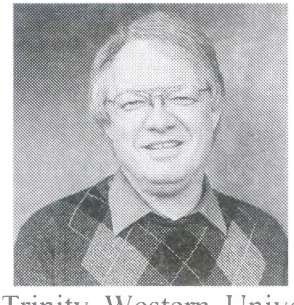
PM Narendra Modi expressed happiness over India's exceptional performance in the IMO and congratulated the team on X (Twitter). PM

Modi even spoke to each of these participants during his radio programme – *Mann Ki Baat*.

The Homi Bhabha Centre for Science Education (HBCSE)-TIFR is the nodal centre for training and selecting students for the international Olympiads. The final team is selected through various stages of the National Olympiad Examination conducted by HBCSE.

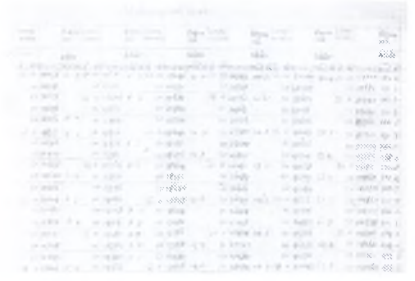
Use of decimal point is 1.5 centuries older than historians thought

Historians have discovered what may be the world's first decimal point, in an ancient manuscript written 150 years before its next known appearance. Scientists recently found the oldest written record of the numeral "0", dating back 500 years earlier than previously thought. Now, it looks like the decimal point is also older than expected.



Previously, the earliest known use of a decimal point was thought to be an astronomical table by the German mathematician *Christopher Clavius* in 1593. Now, a new study has found that the decimal point dates back to the 1440s – about 150 years earlier – first appearing in the writings of Italian mathematician *Giovanni Bianchini*.

Recently, *Glen Van Brummelen*, a professor at Trinity Western University in Canada, has discovered that Bianchini illustrated this system with a decimal point, the first ever. Van Brummelen found that in a manuscript called *Tabulae primi mobilis B*, Bianchini was using numbers with dots in the middle – the first one being 10.4 – and showing how to multiply them, something that was tricky in a base-60 system.



Trigonometrical tables given here from Giovanni Bianchini's 1440s manuscript *Tabulae primi mobilis B*, demonstrats the first known use of the decimal point, where Bianchini breaks numbers into tenths, hundredths and thousandths, denoted with a decimal point after the integer part.

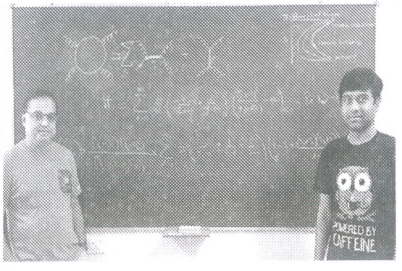
This appears in sections where he tells users how much to add or subtract, to calculate values between entries in the table.

Source: 1. https://newatlas.com/science/decimal-point-history-older/
2. https://phys.org/news/2024-02-decimal-centuries-older-historians-thought.html#google_vignette

3.https://www.popularmechanics.com/science/a60048192/christopher-clavius-decimal-point/

Physicists accidently found a new way to represent π

Our favorite mathematical constant, π, describing the ratio between a circle's circumference and its diameter, has taken on new meaning. Scientists discovered a new series for π through string theory research. Their formula is similar to one by Madhava in the 15th century. Combining the Euler-Beta Function and Feynman Diagram, they created an efficient model, revealing this new π representation. Theoretical work like this can eventually lead to practical applications.



While investigating how string theory can be used to explain certain physical phenomena, Indian Institute of Science (IISc) physicists *Prof. Aninda Sinha* (left) and *Arnab Saha*, a post-doc student (right) discovered a new series representation for π. It provides an easier way to extract π from calculations involved in deciphering processes like the quantum scattering of high-energy particles.

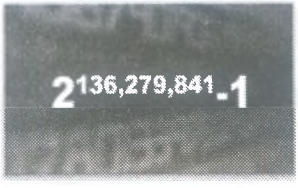
$$\pi = 4 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{1}{n+\lambda} - \frac{4}{2n+1} \right) \left(\frac{(2n+1)^2}{4(n+\lambda)} - n \right)_{n-1}$$

The new formula under a certain limit closely reaches the representation of π suggested by 15th century Indian mathematician *Sangamagrama Madhava*, which was the first ever series for π recorded in history.

Source: 1. https://scitechdaily.com/string-theory-unravels-new-pi-formula- a-quantum-leap-in-mathematics/

2. https://phys.org/news/2024-06-physicists.html#google_vignette

Largest prime number $2^{13,62,79,841} - 1$ of 4,10,24,320 digits discovered



After six-year drought, we now have a new largest known prime number, thanks to an amateur mathematics investigator who used several graphics processing units (GPUs) to search for the number. The Great Internet Mersenne Prime Search (GIMPS) has discovered the largest known prime number $2^{13,62,79,841} - 1$ having 4,10,24,320 decimal digits.

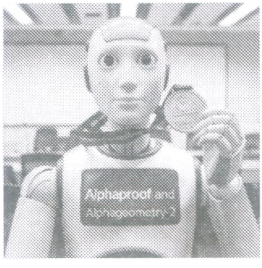
Luke Durant from San Jose, California, along with thousands of volunteers found the prime on 12th Oct. 2024 using free GIMPS software. The new prime number (known as Mersenne primes) is calculated by multiplying together 13,62,79,841 twos, and then subtracting 1. It is only the 52nd known Mersenne prime ever discovered. It is over 16 million digits larger than the previous record prime number. It took nearly 6 years for the GIMPS software to find it after the previous largest known Mersenne prime having 2,48,62,048 digits.

Google DeepMind's new AI systems can now solve complex maths problems

AI models can easily generate essays and other types of text. However, they are nowhere near as good at solving maths problems, which tend to involve logical reasoning – something that is beyond the capabilities of most current AI systems.

But that may finally be changing. Google DeepMind says it has trained two specialized AI systems to solve complex mathematics problems involving advanced reasoning. The systems – called *AlphaProof* and *AlphaGeometry 2* – worked together to successfully solve four out of six problems from the International Mathematical Olympiad 2024 (IMO), a prestigious competition for high school students. They won the equivalent of a silver medal. It is for the first time any AI system has ever achieved such a high success rate on these kinds of problems.

AlphaProof solved two algebra problems and one number theory problem, one of which was the competition's hardest. AlphaGeometry 2 successfully solved a geometry question, but two questions on combinatorics (an area of maths focused on counting and arranging objects) were left unsolved.



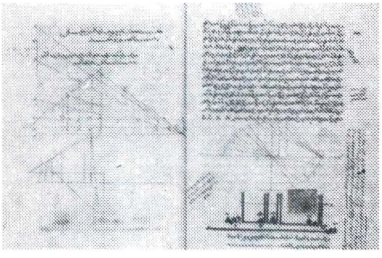
Two renowned mathematicians, *Tim Gowers* and *Joseph Myers*, checked the systems' submissions. They awarded each of their four correct answers full marks (seven out of seven), giving the systems a total of 28 points out of a maximum of 42. A human participant earning this score would be awarded a silver medal and just miss out on gold, the beginning for which starts at 29 points.

This is great progress in the field of machine learning and AI. No such system has been developed until now which could solve problems at this success rate with this level of generality. This is the first time any AI system has been able to achieve a medal-level performance on IMO questions. Creating AI systems that can solve more challenging mathematics problems could pave the way for exciting human-AI collaborations, helping mathematicians to both solve and invent new kinds of problems.

Source: https://www.technologyreview.com/2024/07/25/1095315/google-deepminds-ai-systems-can-now-solve-complex-math-problems/

Scientist rediscover lost works of *Apollonius* in a 17th-century manuscript

Scholars found two lost books by *Apollonius*, the ancient Greek mathematician known as the Great Geometer. The works were found in an Arabic manuscript preserved at the Libraries of Leiden University in the Netherlands. The manuscript had been forgotten in the library, part of a collection acquired by Dutch mathematician *Jacob Golius* in the 17th century.



The rediscovered manuscript contains the lost fifth and seventh books of Apollonius's renowned work, *the Conics*. It introduces fundamental geometric concepts such as hyperbolas, ellipses, and parabolas. These concepts had an impact on the scientific world and are noted for their influence on the ancient mathematics.

Only four of the original eight books of the Conics were available to European scholars during the Renaissance, as the fifth to eighth books were considered lost for centuries. The rediscovery of the lost books reflects the Islamic Golden Age's contributions to preserving knowledge. Islamic scholars preserved and expanded ancient knowledge, which later contributed to the European Renaissance.

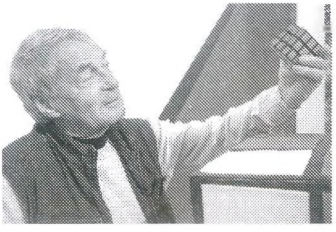
The manuscript is a translation of books five to seven by *Thabit ibn Qurra*, edited by the *Banū Mūsā* brothers. It is accompanied by illustrations and Arabic calligraphy, illuminating the history of mathematics. These manuscripts not only contain mathematical knowledge but also attract attention with calligraphy and geometric illustrations.

Apollonius of Perga, born around 260 BCE in the ancient Greek city of Perga, is known for his pioneering work in geometry. He studied and taught in Alexandria and was one of the greatest mathematicians and geometers of antiquity. Of the 21 works on mathematics, geometry, astronomy, and mechanics that Apollonius wrote, only four have survived.

Source:https://www.jpost.com/archaeology/archaeology-around-the-world/article-841280

The Rubik's Cube turns 50

The Rubik's Cube turns 50 now and, under the ownership of Spin Master, it shows no signs of retirement. Mathematicians and hobbyists have had a half-century of fun exploring the 43 billion billion permutations of Erno Rubik's creation.



Erno Rubik, now 79, is the iconic puzzle toy's creator. Rubik is a Hungarian architect, designer, sculptor and retired professor. Conceived as a mathematical tool to help his students understand three-dimensional movement, the magic cube, as it was initially called, was first pieced together with cubes made from wood and paper held together with rubber bands, glue and paperclips. It was only after the Hungarian sculptor and professor of architecture twisted his completed cube and tried to restore the colored blocks.

In 1974, Rubik was reparing a course on descriptive geometry and toying with the five Platonic solids, he had become especially taken by the cube. For quite a while it never occurred to him that he was creating a puzzle. By about the time of his 30th birthday, in July 1974, he had created the structure, realized its puzzling potential and solved the Cube for the first time. He submitted a patent application in January 1975, and by the end of 1977 the "Magic Cube" had debuted in toy stores in Hungary.

Since its inception in 1974, the Rubik's Cube has spread throughout the pop culture landscape, appearing in movies and TV shows, music videos, comics, video games and museums. The iconic 3-by-3 grid has graced T-shirts, been turned into keychains, inspired architecture and sparked an

entire subgenre of art. Enthusiasts have raced to shrink the cube creator's monthlong solving time, with the current world record standing at a mere 3.13 seconds.

Part of the puzzle's appeal is the complexity that emerges from its simplicity. In its solved state, the Cube's six faces are configured such that all nine sides are the same color. Turning the puzzle scrambles the colors – in total, there are precisely 43,252,003,274,489,856,000 possible positions into which the sides can be permuted.

All the while, the puzzle's essential form – its cubic-ness – remains unchanged. This feature demonstrates group theory, the mathematical study of symmetry. A square has eight symmetries: It can be rotated or reflected four ways each and it's still a square. A plain cube has 48 symmetries. The Rubik's Cube has some 43 quintillion!

Much of the Rubik's Cube staying power is tied to its this complex nature – 43 quintillion combinations, and the human drive to solve problems. Parents and educators have gravitated toward the cube as a way to occupy kids outside the digital world and aid in development of critical thinking skills.

Source:1.https://www.cnbc.com/2024/01/13/rubiks-cube-spin-master-innovating-iconic-puzzle.html

2.https://www.nytimes.com/2024/07/01/science/rubiks-cube-puzzles.html

***Michel Talagrand* wins Abel Prize 2024 for studies of universe's randomness**

A French mathematician and a retired researcher at France's National Center for Scientific Research *Prof. Michel Talagrand* is the recipient of Abel Prize 2024, equivalent of the Nobel prize, for advances in understanding randomness in the universe – the heights of ocean waves crashing on a beach, the weights of babies, the ups and downs of the stock market – work that has found use in mathematical physics and statistics. Dr. Talagrand received 7.5 million Norwegian kroner, or about $7,00,000. In 2019, he had also won the Shaw Prize, another prestigious award.

The citation reads as – "Talagrand is an exceptionally prolific mathematician whose work has transformed probability theory, functional analysis and statistics. His research is characterized by a desire to understand interesting problems at their most fundamental level, building new mathematical theories along the way".

Physicists had previously developed theories on the limits of how randomness behaves, but Talagrand was able to use mathematics to prove these limits. In a sense, things are as simple as could be – whereas mathematical objects can be monstrously complicated. His work deepening the understanding of random phenomena has become essential in today's world. His algorithms are now the basis of weather forecasts and major linguistic models.

Source: 1. https://www.sciencealert.com/abel-prize-winner-who-tamed-randomness-turned-to-math-out-of-necessity

2. https://www.nytimes.com/2024/03/20/science/abel-prize-mathematics-randomness.html

3.https://www.lemonde.fr/en/france/article/2024/03/25/michel-talagrand-an-unlikely-mathematical-superstar-i-wanted-to-take-risks_6650876_7.html

Problem Section

Dear Readers,

As you know, I have added a new series "Problem section" from the Vol.57 (3,4) issue. I had given two problems-one each on Algebra and Geometry. The idea was-and is – that the readers should try to solve these problems and send the answers directly to me. The problems are of somewhat higher level than the school texts and of challenging nature. Even teachers should give it to some students who are interested in solving such problems. We will publish the solutions with their names. But sorry to write, I did not get a single reply!

Anyway, here I am giving the solutions to those problems and also two new problems for the readers to try. Please participate and encourage the students too.

Send your answers with all necessary explanations to me on : mjbhatta@yahoo.com.

- Editor.

Solutions:

Prob.1: If $a^2 + b^2 + c^2 + d^2 - ab - bc - cd + \frac{2}{5} = 0$, then find the value of a+b+c+d; where a,b,c,d $\in$ R.

Think : How will you Start ?

Looking to the structure of the given equation, one may feel that the expansion of $(a + b + c + d)^2$ may help us. But .....?. ab, bc, cd are there but the terms like ad, ac, bd are not but only 'd' is there.

Solution : Let us rearrange the terms to get perfact squares of two terms like this:

$$a^2 + b^2 + c^2 + d^2 - ab - bc - cd + \frac{2}{5} = d$$

$$\Rightarrow a^2 - ab + \frac{b^2}{4} + \frac{3b^2}{4} - bc + \frac{c^2}{3} + \frac{2c^2}{3} - cd + \frac{3d^2}{8} + \frac{5d^2}{8} - d + \frac{2}{5} = 0.$$

$$\Rightarrow \left(a - \frac{b}{2}\right)^2 + \frac{1}{12}(9b^2 - 12bc + 4c^2) + \frac{1}{24}(16c^2 - 24cd + 9d^2)$$

$$+ \frac{1}{40}(25d^2 - 40d + 16) = 0$$

$$\Rightarrow \left(a - \frac{b}{2}\right)^2 + \frac{1}{12}(3b - 2c)^2 + \frac{1}{24}(4c - 3d)^2 + \frac{1}{40}(5d - 4)^2 = 0$$

$$\Rightarrow a - \frac{b}{2} = 0, 3b - 2c = 0, 4c - 3d = 0 \text{ and } 5d - 4 = 0$$

Now $5d - 4 = 0 \Rightarrow d = \frac{4}{5}$

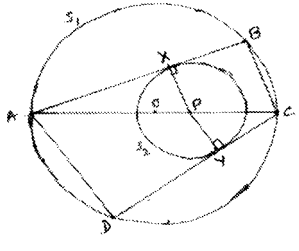
$4c - 3d = 0 \Rightarrow c = 3 \times \frac{4}{5} \times \frac{1}{4} = \frac{3}{5}$

$3b - 2c = 0 \Rightarrow b = 2\left(\frac{3}{5}\right)\left(\frac{1}{3}\right) = \frac{2}{5}$ *and*

$a - \frac{b}{2} = 0 \Rightarrow a = \frac{1}{2}\left(\frac{2}{5}\right) = \frac{1}{5}$

Hence a + $b + c + d = \frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5} = 2.$

Prob.2 :Circle s_1 with centre o and diameter AC is given. P is a point on oc as given in the figure taking P as centre and radius less than PC, another circle s_2 is drawn. Tangents are drawn to s_2 from A and C which intersect the circle s_1 at B and D resp. B and D are on the opposite sider of AC. If AD=a, BC=b and OP=c, find the radius of s_1 in terms of a,b,c.



Think: Looking to the diagram, it is clear that circle s_2 is completely inside circle s_1 and hence tangents to s_2 will be chords in s_1. Also AC is diameter and

hence two semicircles are there to form right angles.
Solu : Let R be the radius of circle s_1 and r be the radius of circle s_2.
$\therefore OA = OC=R$, $AC=2R$
Also $r<PC<PA$ and given $AD=a$, $BC=b$ and $OP=c$, we have
$AP=R+c$ and $PC = R-c$.
The tangent from A to s_2 touches it in x and intersects s_1 in B. Tangent from C to s_2 touches it in Y and intersects s_1 in D.

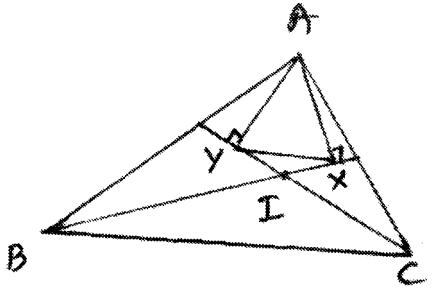
$PX \perp AB$, $PY \perp CD$, $\angle ABC = \angle ADC = 90°$
It is easy to see $\Delta AXP \sim \Delta ABC$ and $\Delta CYP \sim \Delta CDA$. Using ratios for both similarities and eliminating r from the two results, one will get $R= \frac{c(a+b)}{a-b}$.

Further Questions : (1) What will happen if P considers with o ? (2) What will be the result if P is between A and O?

Problems for this issue:
Prob:3 Let a,b,c be positive real numbers such that abc=8. Prove that

$$\frac{ab+4}{a+2}+\frac{bc+4}{b+2}+\frac{ca+4}{c+2} \geq 6.$$

Prob.4 I is the incentre of Δ ABC and X and Y are the feet of perpendiculars from A on BI extended and CI extended respectively. Prove that $\overline{xy} \parallel \overline{BC}$.



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